Continuity of the fundamental operations on distributions having a specified wave front set (with a counter example by Semyon Alesker)

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Abstract

The pull-back, push-forward and multiplication of smooth functions can be extended to distributions if their wave front set satisfies some conditions. Thus, it is natural to investigate the topological properties of these operations between spaces $\mathcal{D}'_\Gamma$ of distributions having a wave front set included in a given closed cone Γ of the cotangent space. As discovered by S. Alesker, the pull-back is not continuous for the usual topology on $\mathcal{D}'_\Gamma$, and the tensor product is not separately continuous. In this paper, a new topology is defined for which the pull-back and push-forward are continuous, the tensor

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and convolution products and the multiplication of distributions are hypocontinuous.

1 Introduction

The motivation of our work comes from the renormalization of QFT in curved space times, indeed the question addressed in this paper cannot be avoided in this context and also the technical results of this paper form the core of the proof that perturbative quantum field theories are renormalizable on curved space times [7, 6].

Since L. Schwartz, we know that the tensor product of distributions is continuous [16, p. 110] and the product of a distribution by a smooth function is hypocontinuous [16, p. 119] (see definition 3.1), although it is not jointly continuous [15].

However, in many applications (for instance the multiplication of distributions), we cannot work with all distributions and we must consider the subsets $\mathcal{D}'_\Gamma$ of distributions whose wave front set [3] is included in some closed subsets Γ of $\dot{T}^*\mathbb{R}^n = \{(x;\xi) \in T^*\mathbb{R}^n\,; \xi \neq 0\}$, where Γ is a *cone* in the sense that $(x;\xi) \in \Gamma$ implies $(x;\lambda\xi) \in \Gamma$ for every $\lambda \in \mathbb{R}_{>0}$. Indeed the spaces $\mathcal{D}'_\Gamma$ are widely used in microlocal analysis because wave front set conditions rule the so-called *fundamental operations on distributions*: multiplication, pull-back and push-forward. The tensor product is also a fundamental operation, but it holds without condition.

Hörmander himself, who introduced the concept of a wave front set [11], equipped $\mathcal{D}'_\Gamma$ with a *pseudo-topology* [11, p. 125], which is not a topology but just a rule describing the convergence of sequences. In particular, when Hörmander writes that the fundamental operations are continuous [12, p. 263], he means "sequentially continuous". And indeed, Hörmander and his followers proved that, under conditions on the wave front set to be described later, the following operators are sequentially continuous: the pull-back of a distribution by a smooth map [12, Thm 8.2.4]; the push-forward of a distribution by a proper map [4, p. 528], the tensor product of two distributions [4, p. 511] and the multiplication of two distributions [4, p. 526].

However, sequential continuity was soon found to be too weak for some applications and Duistermaat [8, p. 18] equipped $\mathcal{D}'_\Gamma$ with a locally convex topology defined in terms of the following seminorms [10, p. 80]:

(i) All the seminorms on $\mathcal{D}'(\mathbb{R}^n)$ for the weak topology: $||u||_\phi = |\langle u, \phi\rangle|$ for all $\phi \in \mathcal{D}(\mathbb{R}^n)$.

(ii) The seminorms $||u||_{N,V,\chi} = \sup_{k\in V}(1+|k|)^N|\widehat{u\chi}(k)|$, where $N \geq 0$, $\chi \in \mathcal{D}(\mathbb{R}^n)$, and $V \in \mathbb{R}^n$ is a closed cone with $\operatorname{supp}\chi \times V \cap \Gamma = \emptyset$.

These seminorms give $\mathcal{D}'_\Gamma$ the structure of a locally convex vector space and the corresponding topology is usually called *Hörmander's topology*. It probably first appeared in the 1970-1971 lecture notes by Duistermaat [8], although the seminorms $||\cdot||_{N,V,\chi}$ are already mentioned by Hörmander [11, p. 128]. The (sequential) convergence in the sense of Hörmander is : a sequence $(u_j) \in \mathcal{D}'_\Gamma$ converges to u in $\mathcal{D}'_\Gamma$ if and only if $||u_j - u||_\phi \to 0$ for every $\phi \in \mathcal{D}(\mathbb{R}^n)$ and $||u_j - u||_{N,V,\chi} \to 0$ for every $\chi \in \mathcal{D}(\Omega)$, every $N \in \mathbb{N}$ and every closed cone V in $\mathbb{R}^n$ such that $\operatorname{supp}\chi \times V \cap \Gamma = \emptyset$. Therefore, it is clear that a sequence converges in the sense of Hörmander if and only if it converges in the sense of Hörmander's topology.

However, for a locally convex space such as $\mathcal{D}'_\Gamma$ (which is not metrizable), sequential continuity and topological continuity are not equivalent. Therefore, when Duistermaat states, after defining the above topology, that the pull-back [8, p. 19], the push-forward [8, p. 20] and the product of distributions [8, p. 21] are continuous, it is not clear whether he means sequential or topological continuity. When investigating this question for applications to valuation theory [1], Alesker discovered a counterexample proving that the tensor product is not separately continuous and the pull-back is not continuous for Hörmander's topology. In other words, this topology is too weak to be useful for these questions.

The purpose of the present paper is to describe Alesker's counterexample and to define a topology for which the fundamental operations have optimal continuity properties: the tensor product is hypocontinuous, the pull-back by a smooth map is continuous, the pull–back by a family of smooth maps depending smoothly on parameters is uniformly continuous, the push-forward by a smooth map is also continuous, the push–forward by a family of smooth maps depending smoothly on parameters is uniformly continuous, the multiplication of distributions and the convolution product are hypocontinuous. Finally, we discuss how the wave front set of distributions on manifolds can be defined in an intrinsic way. In appendices, we prove important technical results concerning the covering of the complement of Γ, the topology of $\mathcal{D}'_\emptyset$ and the fact that the additional seminorms used to define the topology of $\mathcal{D}'_\Gamma$ can be taken to be countable.

The main applications of these results are to replace technical microlocal proofs by classical topological statements [6, 7].

2 Alesker's counterexample

Semyon Alesker discovered the following counter-example

Proposition 2.1. *Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the projection to the first coordinate. Let $\Gamma = \dot{T}^*\mathbb{R}$, so that $\mathcal{D}'_\Gamma(\mathbb{R}) = \mathcal{D}'(\mathbb{R})$, then $f^*\Gamma = \{(x_1, x_2; \xi_1, 0)\}$. We claim that the map $f^* : \mathcal{D}'_\Gamma(\mathbb{R}) \to \mathcal{D}'_{f^*\Gamma}(\mathbb{R}^2)$ is not topologically continuous for the Hörmander topology.*

Note that the general definition of $f^*\Gamma$ is given in Proposition 5.1.

Proof. Let $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\varphi|_{[-1,1]} = 1$. Take $\chi = \varphi \otimes \varphi$, $V = \{(\xi_1, \xi_2) \in \mathbb{R}^2 ; |\xi_1| \le |\xi_2|\}$ and $N = 0$. The intersection of V with $\{(\xi_1, 0) ; \xi_1 \neq 0\}$ is empty because $|\xi_1| \le |\xi_2| = 0$ implies $\xi_1 = \xi_2 = 0$. Therefore, $||\cdot||_{N,V,\chi}$ is a seminorm of $\mathcal{D}'_{f^*\Gamma}$ and, if f^* were continuous, it would be possible to bound $||f^*u||_{N,V,\chi}$ with $\sup_i |\langle u, f_i\rangle|$ for a finite set of $f_i \in \mathcal{D}(\mathbb{R})$ and every $u \in \mathcal{D}'(\mathbb{R})$.

We are going to show that this is not the case. We have

$$||f^*u||_{0,V,\chi} = \sup_{\xi \in V} |\widehat{\varphi u}(\xi_1)|\, |\widehat{\varphi}(\xi_2)| = \sup_{\xi_1} |\widehat{\varphi u}(\xi_1)|\, \omega(\xi_1),$$

where $\omega(\xi_1) = \sup_{|\xi_2| \ge |\xi_1|} |\widehat{\varphi}(\xi_2)|$. It is clear that $\omega(\xi_1) > 0$ everywhere since $\widehat{\varphi}$ is a real analytic function. Thus we should show that the map $\mathcal{D}'(\mathbb{R}) \to \mathbb{R}$ given by $u \mapsto \sup_{\xi \in \mathbb{R}} |\widehat{\varphi u}(\xi)|\omega(\xi)$ is not continuous (for a fixed $\omega > 0$).

If the pull-back were continuous, there would be a finite set $\chi_1, \dots, \chi_t$ of functions in $\mathcal{D}(\mathbb{R})$ such that

$$||f^*u||_{0,V,\chi} \le \sup_{i=1,\dots,t} |\langle u, \chi_i\rangle|.$$

We can find ξ such that the functions $\chi_1, \dots, \chi_t$ and $\varphi(x)e^{-ix\xi}$ are linearly independent. Then there exists $u \in \mathcal{D}'(\mathbb{R})$ such that $\langle u, \chi_i\rangle = 0$ for $i = 1, \dots, t$ and $\widehat{u\varphi}(\xi) = \langle u, \varphi e_\xi\rangle = 1 + 1/\omega(\xi)$, where $e_\xi(x) = e^{-i\xi.x}$. Then, $||f^*u||_{0,V,\chi} = 1 + \omega(\xi)$ and we reach a contradiction. □

Thus, the pull-back is not continuous. Moreover, the same example can be considered as an exterior tensor product $u \to u \boxtimes 1$. This shows that the exterior tensor product is not separately continuous for the Hörmander topology.

3 The normal topology and hypocontinuity

We now modify Hörmander's topology and define what we call the *normal topology* of $\mathcal{D}'_\Gamma$. This is a locally convex topology defined by the same seminorms $||\cdot||_{N,V,\chi}$ as Hörmander's topology, but we replace the seminorms $||\cdot||_\phi$ of the weak topology of $\mathcal{D}'(\mathbb{R}^n)$ by the seminorms $p_B(u) = \sup_{\phi\in B} |\langle u, \phi\rangle|$ (where B runs over the bounded sets of $\mathcal{D}(\Omega)$) of the strong topology of $\mathcal{D}'(\mathbb{R}^n)$. The functional properties of this topology, like completeness, duality, nuclearity, PLS-property, bornologicity, were investigated in detail [5]. As in the case of standard distributions, several operations will not be jointly continuous but only hypocontinuous. Let us recall

Definition 3.1. [17, p. 423] Let E, F and G be topological vector spaces. A bilinear map $f : E \times F \to G$ is said to be *hypocontinuous* if: (i) for every neighborhood W of zero in G and every bounded set $A \subset E$ there is a neighborhood V of zero in F such that $f(A \times V) \subset W$ and (ii) for every neighborhood W of zero in G and every bounded set $B \subset F$ there is a neighborhood U of zero in E such that $f(U \times B) \subset W$.

If E, F and G are locally convex spaces with topologies defined by the families of seminorms $(p_i)_{i\in I}$, $(q_j)_{j\in J}$ and $(r_k)_{k\in K}$, respectively, the definition of hypocontinuity can be translated into the following two conditions: (i) For every bounded set A of E and every seminorm r_k, there is a constant M and a finite set of seminorms $q_{j_1}, \dots, q_{j_n}$ (both depending only on k and A) such that

$$\forall x \in A, r_k\big(f(x,y)\big) \quad\leq\quad M \sup\{q_{j_1}(y), \dots, q_{j_n}(y)\}; \tag{3.1}$$

and (ii) For every bounded set B of F and every seminorm r_k, there is a constant M and a finite set of seminorms $p_{i_1}, \dots, p_{i_n}$ (both depending only on k and B) such that

$$\forall y \in B, r_k\big(f(x,y)\big) \quad\leq\quad M \sup\{p_{i_1}(x), \dots, p_{i_n}(x)\}. \tag{3.2}$$

Equivalently [14, p. 155], we can reformulate hypocontinuity using the concept of equicontinuity [13, p. 200] that is defined as follows :

Definition 3.2. In the general context of a locally convex topological vector space E with seminorms $(p_\alpha)_{\alpha\in A}$. Let E^* be its topological dual, a set H in E^* is called *equicontinuous* if and only if the family of maps $\ell_v := u \in E \longmapsto \langle u, v\rangle \in \mathbb{R}$ is **uniformly** continuous when v runs over the set H.

Hence f is hypocontinuous if for every bounded set A of E and every bounded set B of F the sets of maps $\{f_x\,; x \in A\}$ and $\{f_y\,; y \in B\}$ are equicontinuous, where $f_x : E \to G$ and $f_y : F \to G$ are defined by $f_x(y) = f_y(x) = f(x,y)$.

4 Tensor product of distributions

Let Ω_1 and Ω_2 be open sets in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively, and $(u,v) \in \mathcal{D}'_{\Gamma_1} \times \mathcal{D}'_{\Gamma_2}$, where $\mathcal{D}'_{\Gamma_1} \subset \mathcal{D}'(\Omega_1)$ and $\mathcal{D}'_{\Gamma_2} \subset \mathcal{D}'(\Omega_2)$. Then the tensor product $u \otimes v$ belongs to $\mathcal{D}'_{\Gamma} \subset \mathcal{D}'(\Omega_1 \times \Omega_2)$ where

$$\begin{aligned} \Gamma &= \big(\Gamma_1 \times \Gamma_2\big) \cup \big((\Omega_1 \times \{0\}) \times \Gamma_2\big) \cup \big(\Gamma_1 \times (\Omega_2 \times \{0\})\big) \\ &= (\Gamma_1 \cup \{\underline{0}\}_1) \times (\Gamma_2 \cup \{\underline{0}\}_2) \setminus \{(\underline{0},\underline{0})\}, \end{aligned}$$

$\{\underline{0}\}_1$ means $\Omega_1 \times \{0\}$, $\{\underline{0}\}_2$ means $\Omega_2 \times \{0\}$ and $\{\underline{0},\underline{0}\}$ means $(\Omega_1 \times \Omega_2) \times \{0,0\}$. Our goal in this section is to show that the tensor product is hypocontinuous for the normal topology. We denote by $(z;\zeta)$ the coordinates in $T^*(\Omega_1 \times \Omega_2)$, where $z = (x,y)$ with $x \in \Omega_1$ and $y \in \Omega_2$, $\zeta = (\xi,\eta)$ with $\xi \in \mathbb{R}^{d_1}$ and $\eta \in \mathbb{R}^{d_2}$. We also denote $d = d_1 + d_2$, so that $\zeta \in \mathbb{R}^d$.

Lemma 4.1. *The seminorms of the strong topology of $\mathcal{D}'(\mathbb{R}^d)$ and the family of seminorms:*

$$\|t_1 \otimes t_2\|_{N,V,\varphi_1\otimes\varphi_2} = \sup_{\zeta \in V} (1+|\zeta|)^N |\widehat{t_1\varphi_1}(\xi)|\, |\widehat{t_2\varphi_2}(\eta)|, \tag{4.1}$$

where $\zeta = (\xi,\eta)$, $(\varphi_1,\varphi_2) \in \mathcal{D}(\Omega_1) \times \mathcal{D}(\Omega_2)$ and $V \subset \mathbb{R}^d$, are such that $(\text{supp } (\varphi_1 \otimes \varphi_2) \times V) \cap \Gamma = \emptyset$*, are a fundamental system of seminorms for the normal topology of $\mathcal{D}'_\Gamma$.*

Proof. We use the following lemma [10, p. 80]

Lemma 4.2. *Let Ω be an open set in $\mathbb{R}^n$. If we have a family, indexed by $\alpha \in A$, of $\chi_\alpha \in \mathcal{D}(\Omega)$ and of closed cones $V_\alpha \subset (\mathbb{R}^n \backslash \{0\})$ such that* $(\text{supp}\,\chi_\alpha \times V_\alpha) \cap \Gamma = \emptyset$ *and*

$$\Gamma^c \;=\; \bigcup_{\alpha \in A} \{(x,\xi) \in \dot{T}^*\Omega\,; \chi_\alpha(x) \neq 0, \xi \in \mathring{V}_\alpha\},$$

then the topology of $\mathcal{D}'_\Gamma$ is already defined by the strong topology of $\mathcal{D}'(\Omega)$ and the seminorms $||\cdot||_{N,V_\alpha,\chi_\alpha}$.

It is clear that the family indexed by $\varphi_1 \otimes \varphi_2$ and V such that supp $(\varphi_1 \otimes \varphi_2) \times V \cap \Gamma = \emptyset$ satisfies the hypothesis of the lemma. $\square$

To establish the hypocontinuity of the tensor product, we consider an arbitrary bounded set $B \subset \mathcal{D}'_{\Gamma_1}(\Omega_1)$ and, according to eq. (3.1), we must show that, for every seminorm r_k of $\mathcal{D}'_{\Gamma}(\Omega_1 \times \Omega_2)$, there is a constant M and a finite number of seminorms q_j such that $r_k(u \otimes v) \leqslant M \sup_j q_j(v)$ for every $u \in B$ and every $v \in \mathcal{D}'_{\Gamma_2}(\Omega_2)$. By Schwartz' theorem [16, p. 110] we already know that this is true for every seminorm r_k of the strong topology of $\mathcal{D}'_{\Gamma}(\Omega_1 \times \Omega_2)$. It remains to show it for every $|| \cdot ||_{N,V,\varphi_1 \otimes \varphi_2}$. This will be done by first defining a suitable partition of unity on $\Omega_1 \times \Omega_2$ and its corresponding cones. Then, this partition of unity will be used to bound the seminorms by standard microlocal techniques.

Lemma 4.3. *Let* Γ_1, Γ_2 *be closed cones in* $\dot{T}^*\Omega_1$ *and* $\dot{T}^*\Omega_2$*, respectively. Set* $\Gamma = (\Gamma_1 \cup \{\underline{0}\}) \times (\Gamma_2 \cup \{\underline{0}\}) \setminus \{(\underline{0}, \underline{0})\} \subset \dot{T}^*\mathbb{R}^d$*. Then for all closed cones* $V \subset \mathbb{R}^d$ *and* $\chi \in \mathcal{D}(\Omega_1 \times \Omega_2)$ *such that* $(\operatorname{supp} \chi \times V) \cap \Gamma = \emptyset$*, there exist a partition of unity* $(\psi_{j1} \otimes \psi_{j2})_{j \in J}$ *of* $\Omega_1 \times \Omega_2$*, which is finite on* $\operatorname{supp} \chi$*, and a family of closed cones* $(W_{j1} \times W_{j2})_{j \in J}$ *in* $(\mathbb{R}^{d_1} \setminus \{0\}) \times (\mathbb{R}^{d_2} \setminus \{0\})$ *such that*

$$\begin{aligned} (4.2) \qquad & (\operatorname{supp} \psi_{j1} \times W_{j1}^c) \cap \Gamma_1 &=& \ (\operatorname{supp} \psi_{j2} \times W_{j2}^c) \cap \Gamma_2 = \emptyset, \\ (4.3) \qquad & V \cap ((W_{j1} \cup \{0\}) \times (W_{j2} \cup \{0\})) &=& \ \emptyset, \\ & \textit{if } \operatorname{supp} \chi \cap \operatorname{supp} (\psi_{j1} \otimes \psi_{j2}) \neq \emptyset. \end{aligned}$$

Proof. — We first set some notation. For any $D \in \mathbb{N}$, with the identification $T^*\mathbb{R}^D \simeq \mathbb{R}^D \oplus (\mathbb{R}^D)^*$, we denote by $\underline{\pi} : T^*\mathbb{R}^D \longrightarrow \mathbb{R}^D$ the projection onto the first factor and by $\overline{\pi} : T^*\mathbb{R}^D \longrightarrow (\mathbb{R}^D)^*$ the projection on the second factor. We use the distance d_∞ on $\mathbb{R}^D$ (or $(\mathbb{R}^D)^*$) defined by $d_\infty(u, v) := \sup_{1 \leq i \leq D} |u^i - v^i|$. For $u \in \mathbb{R}^D$ and $r \geq 0$ we then set $\overline{B}(u, r) = \{v \in \mathbb{R}^D; d_\infty(u, v) \leq r\}$ and, for any subset $Q \subset \mathbb{R}^D$, $Q_{,r} := \{v \in \mathbb{R}^D; d_\infty(v, Q) \leq r\}$. We note that, for any pair of sets $Q_1 \subset \mathbb{R}^{d_1}$ and $Q_2 \subset \mathbb{R}^{d_2}$, $(Q_1 \times Q_2)_{,r} = Q_{1,r} \times Q_{2,r}$ (in particular, if $(x, y) \in \Omega_1 \times \Omega_2$, $\overline{B}((x, y), r) = \overline{B}(x, r) \times \overline{B}(y, r)$). Lastly for any closed conic subset $W \subset (\mathbb{R}^D)^* \setminus \{0\}$, we set $\overline{W} := W \cup \{0\}$ for short and $UW := S^{D-1} \cap W$. Similarly if Γ is a conic subset of $T^*\mathbb{R}^D$, we set $U\Gamma = (\mathbb{R}^D \times S^{D-1}) \cap \Gamma$ and $\overline{\Gamma} = \Gamma \cup \underline{0} \subset T^*\mathbb{R}^D$ where $\underline{0}$ is the zero section of $T^*\mathbb{R}^D$.

We will prove that there exists a family of open balls $(B_{j1} \times B_{j2})_{j \in J}$ that covers $\Omega_1 \cap \Omega_2$, which is finite over any compact subset of $\Omega_1 \times \Omega_2$ and in particular on $\operatorname{supp} \chi$ and such that $(\overline{B_{j1}} \times W_{j1}^c) \cap \Gamma_1 = (\overline{B_{j2}} \times W_{j2}^c) \cap \Gamma_2 = \emptyset$ and that $V \cap (\overline{W}_{j1} \times \overline{W}_{j2}) = \emptyset$, if $\operatorname{supp} \chi \cap (\overline{B}_{j1} \times \overline{B}_{j2}) \neq \emptyset$. The conclusion of the lemma will then follow by constructing a partition of unity $(\psi_{j1} \otimes \psi_{j2})_{j \in J}$

such that $\operatorname{supp}\psi_{j1} = \overline{B}_{j1}$ and $\operatorname{supp}\psi_{j2} = \overline{B}_{j2}$, $\forall j \in J$, by using standard arguments.

Step 1. If $(\operatorname{supp}\chi \times V) \cap \Gamma = \emptyset$, then there exists some $\delta > 0$ such that $d_\infty(\operatorname{supp}\chi \times UV, U\Gamma) \geq 4\delta$. Consider $K := (\operatorname{supp}\chi)_{,\delta}$, we then note that $d_\infty(K \times UV, U\Gamma) \geq 3\delta$. Without loss of generality, we can assume that δ has been chosen so that $K \subset \Omega_1 \times \Omega_2$. Obviously $\Omega_1 \times \Omega_2$ is covered by $(B((x,y),\delta))_{(x,y)\in\Omega_1\times\Omega_2}$. Moreover all balls $B((x,y),\delta)$ are contained in K if $(x,y) \in \operatorname{supp}\chi$ and $\operatorname{supp}\chi$ is covered by the subfamily $(B((x,y),\delta))_{(x,y)\in\operatorname{supp}\chi}$. Since $\operatorname{supp}\chi$ is compact we can thus extract a countable family of balls $(B_i)_{i\in I} = (B_{i1} \times B_{i2})_{i\in I}$ which covers $\Omega_1 \times \Omega_2$ and which is finite over $\operatorname{supp}\chi$.

We now set $\gamma := \overline{\pi}(\underline{\pi}^{-1}(K) \cap \Gamma)$ and $U\gamma := \overline{\pi}(\underline{\pi}^{-1}(K) \cap U\Gamma)$ and we estimate the distance of $U\gamma$ to UV:

$$\begin{aligned} d_\infty[U\gamma, UV] &= \inf_{\xi\in\overline{\pi}(\underline{\pi}^{-1}(K)\cap U\Gamma)} \inf_{\eta\in UV} d_\infty(\xi,\eta) \\ &= \inf_{(u,\xi)\in U\Gamma; u\in K} \inf_{(v,\eta)\in K\times UV} d_\infty(\xi,\eta) \\ &= \inf_{(u,\xi)\in U\Gamma; u\in K} \inf_{(v,\eta)\in K\times UV} d_\infty((u,\xi),(v,\eta)), \end{aligned}$$

where the last equality is due to the fact that one can choose $v = u$ in the minimization. We deduce that, by removing the constraint $u \in K$ in the minimization,

$$\begin{aligned} d_\infty[U\gamma, UV] &\geq \inf_{(u,\xi)\in U\Gamma} \inf_{(v,\eta)\in K\times UV} d_\infty((u,\xi),(v,\eta)) \\ &= d_\infty(K \times UV, U\Gamma) \geq 3\delta. \end{aligned}$$

Step 2. Since γ and V are cones, the previous inequality implies $d_\infty(\xi, V) \geq 2\|\xi\|\delta$ for every $\xi \in \gamma$. For any $i \in I$ such that the ball B_i is centered at a point in $\operatorname{supp}\chi$, the inclusion $\overline{B_i} \subset K$ implies $\overline{\pi}(\underline{\pi}^{-1}(\overline{B_i}) \cap \Gamma) \subset \gamma$. We hence have also

$$\forall\xi \in \overline{\pi}(\underline{\pi}^{-1}(\overline{B_i}) \cap \Gamma) \quad d_\infty(\xi, V) \geq 2\|\xi\|\delta. \tag{4.4}$$

We now set $\overline{W}_{i1} := \{\xi_1 \in (\mathbb{R}^{d_1})^*; d_\infty(\xi_1, \overline{\pi}(\underline{\pi}^{-1}(\overline{B_{i1}}) \cap \Gamma_1)) \leq \|\xi_1\|\delta\}$, $\overline{W}_{i2} := \{\xi_2 \in (\mathbb{R}^{d_2})^*; d_\infty(\xi_2, \overline{\pi}(\underline{\pi}^{-1}(\overline{B_{i2}}) \cap \Gamma_2)) \leq \|\xi_2\|\delta\}$ and $W_{i1} := \overline{W}_{i1} \setminus \{0\}$, $W_{i2} := \overline{W}_{i2} \setminus \{0\}$. By the definition of W_{i1}, $W_{i1}^c \cap \overline{\pi}(\underline{\pi}^{-1}(\overline{B_{i1}}) \cap \Gamma_1) = \emptyset$, which is equivalent to $(\overline{B}_{i1} \times W_{i1}^c) \cap \Gamma_1 = \emptyset$. Similarly $(\overline{B}_{i2} \times W_{i1}^c) \cap \Gamma_2 = \emptyset$.

On the other hand, since

$$\begin{aligned} \overline{\pi}(\underline{\pi}^{-1}(\overline{B_{i1}}) \cap \overline{\Gamma_1}) \times \overline{\pi}(\underline{\pi}^{-1}(\overline{B_{i2}}) \cap \overline{\Gamma_2}) &= \overline{\pi}[\underline{\pi}^{-1}(\overline{B_{i1}} \times \overline{B_{i2}}) \cap (\overline{\Gamma_1} \times \overline{\Gamma_2})] \\ &= \overline{\pi}[\underline{\pi}^{-1}(\overline{B_i}) \cap \overline{\Gamma}], \quad \overline{B_i} = \overline{B_{i1}} \times \overline{B_{i2}} \end{aligned}$$

because

$$\{\xi_1; \exists(x_1;\xi_1) \in \overline{\Gamma_1}, x_1 \in \overline{B_{i1}}\} \times \{\xi_2; \exists(x_2;\xi_2) \in \overline{\Gamma_2}, x_2 \in \overline{B_{i2}}\}$$

$$= \{(\xi_1,\xi_2); \exists(x_1,x_2;\xi_1,\xi_2) \in \overline{\Gamma_1} \times \overline{\Gamma_2}, (x_1,x_2) \in \overline{B_{i1}} \times \overline{B_{i2}}\}$$

we also have

$$\begin{array}{rcl} \overline{W}_{i1} \times \overline{W}_{i2} & = & \{(\xi_1,\xi_2) \in (\mathbb{R}^d)^*; d_\infty[(\xi_1,\xi_2), \overline{\pi}(\underline{\pi}^{-1}(\overline{B_i}) \cap \Gamma)] \\ & \leq & \sup(\|\xi_1\|, \|\xi_2\|)\delta\}. \end{array}$$

Hence by (4.4), we deduce that $\overline{W}_{i1} \times \overline{W}_{i2}$ does not meet V. □

In the rest of the paper, we may identify abusively $\mathbb{R}^d$ and $(\mathbb{R}^d)^*$. We also introduce the notation $e_\zeta(x,y) = e^{i(\xi.x+\eta.y)}$ where $\zeta = (\xi,\eta)$. To estimate $||u\otimes v||_{N,V,\varphi_1\otimes\varphi_2}$, we use Lemma 4.3 to find a partition of unity $(\psi_{j1}\otimes\psi_{j2})_{j\in J}$ which is finite on $\mathrm{supp}\,(\varphi_1\otimes\varphi_2)$ to write

$$\begin{array}{rcl} \widehat{u\varphi_1}(\xi)\widehat{v\varphi_2}(\eta) = \mathcal{F}(u\varphi_1 \otimes v\varphi_2)(\zeta) & = & \langle u\otimes v, (\varphi_1\otimes\varphi_2)e_\zeta\rangle \\ = \displaystyle\sum_j \langle u\otimes v, (\varphi_1\psi_{j1}\otimes\varphi_2\psi_{j2})e_\zeta\rangle & = & \displaystyle\sum_j \widehat{u\varphi_1\psi_{j1}}(\xi)\,\widehat{v\varphi_2\psi_{j2}}(\eta). \end{array}$$

Therefore $||u\otimes v||_{N,V,\varphi_1\otimes\varphi_2} \leqslant \sum_j ||u\otimes v||_{N,V,\varphi_1\psi_{j1}\otimes\varphi_2\psi_{j2}}$, where the sum over j is finite. Each seminorm on the right hand side is bounded by the following lemma.

Lemma 4.4. *Let Γ_1, Γ_2 and Γ be closed cones as in the previous lemma, $\psi_1 \in \mathcal{D}(\Omega_1)$ $\psi_2 \in \mathcal{D}(\Omega_2)$ such that* $(\mathrm{supp}\,(\psi_1\otimes\psi_2)\times V)\cap\Gamma = \emptyset$ *and closed cones W_1 and W_2 in $\mathbb{R}^{d_1}\backslash\{0\}$ and $\mathbb{R}^{d_2}\backslash\{0\}$ such that*

$$(W_1\cup\{0\})\times(W_2\cup\{0\})\cap V = \emptyset, \tag{4.5}$$

$$(\mathrm{supp}\,\psi_k \times W_k^c)\cap\Gamma_k = \emptyset, \text{ for } k=1,2. \tag{4.6}$$

Then, for every bounded set $A \subset \mathcal{D}'_{\Gamma_1}$ and every integer N, there are constants m, M_1, M_2 and a bounded set $B \subset \mathcal{D}(K)$, where K is an arbitrary compact neighborhood of $\mathrm{supp}\,\psi_2$*, such that*

$$||t_1\otimes t_2||_{N,V,\psi_1\otimes\psi_2} \leq M_1||t_2||_{N,C_\beta,\psi_2} + M_2||t_2||_{N+m,C_\beta,\psi_2} + p_B(t_2),$$

for every $t_1 \in A$ and $t_2 \in \mathcal{D}'_{\Gamma_2}$, where C_β is an arbitrary conic neighborhood of W_2 with compact base and p_B is a seminorm of the strong topology of $\mathcal{D}'(\mathbb{R}^{d_2})$.

Proof. We want to calculate

$$||t_1 \otimes t_2||_{N,V,\psi_1\otimes\psi_2} = \sup_{\zeta\in V}(1+|\zeta|)^N|\mathcal{F}(t_1\psi_1\otimes t_2\psi_2)(\zeta)|.$$

We denote $u = t_1\psi_1$, $v = t_2\psi_2$ and $I = \widehat{u\otimes v}$. From $e_{(\xi,\eta)} = e_\xi \otimes e_\eta$ we find that $I(\xi,\eta) = \langle t, e_{(\xi,\eta)}\rangle = \langle u\otimes v, e_\xi\otimes e_\eta\rangle = \langle u, e_\xi\rangle\langle v, e_\eta\rangle = \widehat{u}(\xi)\widehat{v}(\eta)$. By the shrinking lemma we can slightly enlarge W_1 and W_2 to closed cones having the same properties. Thus, there are two homogeneous functions of degree zero α and β on $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively, which are smooth except at the origin, non-negative and bounded by 1, such that: (i) $\alpha|_{W_1\cup\{0\}} = 1$ and $\beta|_{W_2\cup\{0\}} = 1$; (ii) $(\operatorname{supp}\alpha\times\operatorname{supp}\beta)\cap V = \emptyset$; (iii) $(\operatorname{supp}\psi_1\times\operatorname{supp}(1-\alpha))\cap\Gamma_1 = \emptyset$; (iv) $(\operatorname{supp}\psi_2\times\operatorname{supp}(1-\beta))\cap\Gamma_2 = \emptyset$. We can write $I = I_1+I_2+I_3+I_4$ where (recalling that $\zeta = (\xi,\eta)$)

$$\begin{aligned}
I_1(\zeta) &= \alpha(\xi)\widehat{u}(\xi)\beta(\eta)\widehat{v}(\eta),\\
I_2(\zeta) &= \alpha(\xi)\widehat{u}(\xi)(1-\beta)(\eta)\widehat{v}(\eta),\\
I_3(\zeta) &= (1-\alpha)(\xi)\widehat{u}(\xi)\beta(\eta)\widehat{v}(\eta),\\
I_4(\zeta) &= (1-\alpha)(\xi)\widehat{u}(\xi)(1-\beta)(\eta)\widehat{v}(\eta).
\end{aligned}$$

The term $I_1(\zeta) = 0$ because, by condition (ii) $\alpha(\xi)\beta(\eta) = 0$ for $(\xi,\eta)\in V$. Condition (iii) implies that

$$|(1-\alpha)(\xi)\widehat{u}(\xi)| \leq \sup_{\xi\in C_\alpha}|\widehat{t_1\psi_1}(\xi)| \leq (1+|\xi|)^{-N}||t_1||_{N,C_\alpha,\psi_1},$$

where $\xi\in C_\alpha = \operatorname{supp}(1-\alpha)$. This gives us, with $C_\beta = \operatorname{supp}(1-\beta)$,

$$\begin{aligned}
|I_4(\zeta)| &\leq (1+|\xi|)^{-N}(1+|\eta|)^{-N}||t_1||_{N,C_\alpha,\psi_1}||t_2||_{N,C_\beta,\psi_2}\\
&\leq (1+|\zeta|)^{-N}||t_1||_{N,C_\alpha,\psi_1}||t_2||_{N,C_\beta,\psi_2},
\end{aligned}$$

because $1+|(\xi,\eta)| \leq 1+|\xi|+|\eta| \leq (1+|\xi|)(1+|\eta|)$. Since the set A is bounded in $\mathcal{D}'_{\Gamma_1}$ there is a constant $M_1 = \sup_{t_1\in A}\|t_1\|_{N,C_\alpha,\psi_1}$ such that $|I_4(\zeta)| \leq (1+|\zeta|)^{-N}M_1||t_2||_{N,C_\beta,\psi_2}$.

To estimate I_2, we use the fact that, $u = t_1\psi_1$ being a compactly supported distribution there is an integer m such that, for all $t_1\in A$,

$$|\alpha(\xi)\widehat{u}(\xi)| \leq |\widehat{u}(\xi)| \leq (1+|\xi|)^m||\theta^{-m}\widehat{u}||_{L^\infty}.$$

As for the estimate of I_4, we get $|(1-\beta)(\eta)\widehat{v}(\eta)| \leq (1+|\eta|)^{-N-m}||t_2||_{N+m,C_\beta,\psi_2}$. The set $\overline{\{\zeta\in\operatorname{supp}\alpha\times C_\beta\,;|\zeta|=1\}\cap V}$ is **compact** and avoids the set of all elements of the form $\zeta = (\xi,0), \xi\in\operatorname{supp}\alpha\setminus\{0\}$. Otherwise, we would find some sequence $(\xi_n, \eta_n) \to (\xi, 0) \in ((\mathrm{supp}\ \alpha \times \{0\}) \cap V) \subset ((\mathrm{supp}\ \alpha \times \mathrm{supp}\ \beta) \cap V)$ which contradicts the condition (ii). Let $\epsilon > 0$ be the smallest value of $|\eta|$ in this set. Then, the functions α and β being homogeneous of degree zero, $\mathrm{supp}\, \alpha \times C_\beta \cap V$ is a cone in $\mathbb{R}^d$ and $|\eta|/|\zeta| \geq \epsilon$ for all $\zeta = (\xi, \eta)$ in the set $\mathrm{supp}\, \alpha \times C_\beta \cap V$. Thus, $(1+|\eta|)^{-N-m} \leq \epsilon^{-N-m}(1+|(\xi,\eta)|)^{-N-m}$ and $|I_2(\zeta)| \leq ||\theta^{-m}\widehat{t_1\psi_1}||_{L^\infty}||t_2||_{N+m,C_\beta,\psi_2}\epsilon^{-N-m}(1+|\zeta|)^{-N}$, for every $\zeta \in V$, because $|\xi| \leq |(\eta, \xi)|$. We now prove an intermediate lemma:

Lemma 4.5. *Let Ω be an open set of $\mathbb{R}^d$ and B a bounded set in $\mathcal{D}'(\Omega)$, then for every $\chi \in \mathcal{D}(\Omega)$ there exist an integer M and a constant C (both depending only on B and on an arbitrary relatively compact open neighborhood of* $\mathrm{supp}\, \chi$*) such that*

$$\sup_{u\in B}\sup_{\xi\in\mathbb{R}^n}(1+|\xi|)^{-M}|\widehat{u\chi}(\xi)| < 2^M C\,\mathrm{Vol}(K)\,\pi_{M,K}(\chi),$$

where $\pi_{m,K}(\chi) = \sup_{x\in K, |\alpha|\leqslant M}|\partial^\alpha\chi(x)|$ *and for* $K = \mathrm{supp}\, \chi$.

Proof. Let Ω_0 be a relatively compact open neighborhood of $K = \mathrm{supp}\, \chi$. According to Schwartz [16, p. 86], for any bounded set B in $\mathcal{D}'(\Omega)$, there is an integer M (depending only on B and Ω_0) such that every $u \in B$ can be expressed in Ω_0 as $u = \partial^\alpha f_u$ for $|\alpha| \leq M$, where f_u is a continuous function. Moreover, there is a constant C (depending only on B and Ω_0) such that $|f_u(x)| \leq C$ for all $x \in \Omega_0$ and $u \in B$. Thus,

$$\begin{aligned}\widehat{u\chi}(\xi) &= \int_{\Omega_0} e^{-i\xi\cdot x}\chi(x)\partial^\alpha f_u(x)dx = (-1)^{|\alpha|}\int_{\Omega_0} f_u(x)\partial^\alpha\big(e^{-i\xi\cdot x}\chi(x)\big)dx\\ &= (-1)^{|\alpha|}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}(i\xi)^\beta\int_{\Omega_0} f_u(x)e^{-i\xi\cdot x}\partial^{\beta-\alpha}\chi(x)dx.\end{aligned}$$

By using $|(i\xi)^\beta| \leq (1+|k|)^M$ if $|\beta| \leq M$ we obtain

$$\begin{aligned}(1+|\xi|)^{-M}|\widehat{u\chi}(\xi)| &\leq \sup_{|\alpha|\leq M}\sum_{\beta\leq\alpha}\binom{\alpha}{\beta}\Big|\int_{\Omega_0} f_u(x)e^{-i\xi\cdot x}\partial^{\beta-\alpha}\chi(x)dx\Big|\\ &\leq 2^M C\,\mathrm{Vol}(K)\,\pi_{M,K}(\chi).\end{aligned}$$

□

By lemma 4.5, $||\theta^{-m}\widehat{t_1\psi_1}||_{L^\infty}$ is uniformly bounded for $t_1 \in A$ by a constant $M_2' = \sup_{t_1\in A}||\theta^{-m}\widehat{t_1\psi_1}||_{L^\infty}, \theta = 1+|\xi|$. Therefore, there is a constant $M_2 = M_2'\epsilon^{-N-m}$ such that, for every $t_1 \in A$ and every $t_2 \in \mathcal{D}'_{\Gamma_2}$, $|I_2(\zeta)| \leq M_2||t_2||_{N+m,C_\beta,\psi_2}$.

The term I_3 is treated differently because we want to get the following result: for every bounded set A in $\mathcal{D}'_{\Gamma_1}$ and every seminorm $||\cdot||_{N,V,\chi}$, there is a bounded set $B \in \mathcal{D}(\Omega_2)$ such that for all $\zeta \in V, I_3(\zeta) \leq p_B(t_2)(1+|\zeta|)^{-N}$ for every $t_2 \in \mathcal{D}'_{\Gamma_2}$. This special form of eq. (3.1) is possible because the union of bounded sets is a bounded set and the multiplication of a bounded set by a positive constant M is a bounded set.

We write $I_3(\zeta) = \langle t_2, f_\zeta\rangle$, where $f_{(\xi,\eta)}(y) = (1-\alpha)(\xi)\widehat{u}(\xi)\beta(\eta)\psi_2(y)e_\eta(y)$ and we must show that the set $B = \{(1+|\zeta|)^N f_\zeta\,; \zeta \in V\}$ is a bounded set of $\mathcal{D}(\Omega_2)$. A subset B of $\mathcal{D}(\Omega_2)$ is bounded if and only if there is a compact set K and a constant M_n for every integer n such that $\operatorname{supp} f \subset K$ and $\pi_{m,K}(f) \leq M_n$ for every $f \in B$. All f_ζ are supported on $\operatorname{supp}\psi_2$ and are smooth functions because ψ_2 and e_η are smooth. We have to prove that, if t_1 runs over a bounded set of $\mathcal{D}'_{\Gamma_1}$, then there are constants M_n such that $\pi_{n,K}(f_\zeta) \leq M_n$ for all $\zeta \in V$, where K is a compact neighborhood of $\operatorname{supp}\psi_2$. We start from

$$\pi_{n,K}(f_{(\xi,\eta)}) \;=\; \big|(1-\alpha)(\xi)\widehat{u}(\xi)\beta(\eta)\big|\pi_{n,K}(\psi_2 e_\eta).$$

We notice that $\pi_{n,K}(\psi_2 e_\eta) \leq 2^n \pi_{n,K}(\psi_2)\pi_{n,K}(e_\eta)$ and that $\pi_{n,K}(e_\eta) \leq |\eta|^n$. As for the estimate of I_2, we have $|(1-\alpha)(\xi)\widehat{u}(\xi)| \leq (1+|\xi|)^{-N-n}||t_1||_{N+n,C_\alpha,\psi_1}$ because $(\operatorname{supp}\varphi_1 \times \operatorname{supp}(1-\alpha)) \cap \Gamma_1 = \emptyset$ and $(1+|\xi|)^{-N-n} \leq \epsilon^{-N-n}(1+|(\xi,\eta)|)^{-N-n}$ for some ε because $(\xi,\eta) \in (V \cap \operatorname{supp}(1-\alpha) \times \operatorname{supp}\beta)$. Therefore

$$\pi_{n,K}(f_\zeta) \;\leq\; ||t_1||_{N+n,C_\alpha,\psi_1} 2^n \pi_{n,K}(\psi_2)\epsilon^{-N-n}(1+|\zeta|)^{-N},$$

because $|\eta|^n(1+|(\xi,\eta)|)^{-n} \leq 1$. If t_1 belongs to a bounded set A of $\mathcal{D}'_{\Gamma_1}$, then for each N $||t_1||_{N,C_\alpha,\psi_1}$ is uniformly bounded. The estimate of I_3 is finally

$$|I_3(\zeta)| \;\leq\; p_B(t_2)(1+|\zeta|)^{-N}.$$

□

For each j, the conditions of the lemma hold if we put $\psi_1 = \varphi_1\psi_{j1}$, $\psi_2 = \varphi_2\psi_{j2}$, $W_1 = W_{j1}$ and $W_2 = W_{j2}$. Thus, for every bounded set A in $\mathcal{D}'_{\Gamma_1}$, every $u \in A$ and every $v \in \mathcal{D}'_{\Gamma_2}$ we have

$$\begin{aligned}||u\otimes v||_{N,V,\varphi_1\otimes\varphi_2} &\leq \sum_j ||u\otimes v||_{N,V,\varphi_1\psi_{j1}\otimes\varphi_2\psi_{j2}}\\ &\leq \sum_j M_{1j}||v||_{N,C_{\beta_j},\varphi_2\psi_{j2}} + M_2||v||_{N+m,C_{\beta_j},\varphi_2\psi_{j2}} + p_{B_j}(v).\end{aligned}$$

Since the sum over j is finite, this means that the family of maps $u \times v \mapsto u \otimes v$, where $u \in A$, is equicontinuous for any bounded set $A \subset \mathcal{D}'_{\Gamma_1}$. Because of the symmetry of the problem, we can prove similarly that the family of maps $u \times v \mapsto u \otimes v$, where $v \in B$, is equicontinuous for any bounded set $B \subset \mathcal{D}'_{\Gamma_2}$. Finally, we have proved

Theorem 4.6. *Let $\Omega_1 \subset \mathbb{R}^{d_1}$, $\Omega_2 \subset \mathbb{R}^{d_2}$ be open sets, $\Gamma_1 \in \dot{T}^*\Omega_1$, $\Gamma_2 \in \dot{T}^*\Omega_2$ be closed cones and*

$$\Gamma = \big(\Gamma_1 \times \Gamma_2\big) \cup \big((\Omega_1 \times \{0\}) \times \Gamma_2\big) \cup \big(\Gamma_1 \times (\Omega_2 \times \{0\})\big).$$

Then, the tensor product $(u, v) \mapsto u \otimes v$ is hypocontinuous from $\mathcal{D}'_{\Gamma_1} \times \mathcal{D}'_{\Gamma_2}$ to $\mathcal{D}'_{\Gamma}$, in the normal topology.

5 The pull-back

The purpose of this section is to prove

Proposition 5.1. *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and Γ a closed cone in $\dot{T}^*\Omega_2$. Let $f : \Omega_1 \to \Omega_2$ be a smooth map such that $N_f \cap \Gamma = \emptyset$, where $N_f = \{(f(x); \eta) \in \Omega_2 \times \mathbb{R}^n\,; \eta \circ df_x = 0\}$ and $f^*\Gamma = \{(x; \eta \circ df_x)\,; (f(x); \eta) \in \Gamma\}$, where*

$$\begin{aligned} \eta \circ df_x &:= \sum_{j=1}^{d_2} \eta_j d(y^j \circ f)_x = \sum_{j=1}^{d_2} \eta_j dy^j \circ df_x \\ &= \sum_{j=1}^{d_2} \eta_j df_x^j = \sum_{j=1}^{d_2} \sum_{i=1}^{d_1} \eta_j \frac{\partial f^j}{\partial x^i} dx^i. \end{aligned}$$

Then, the pull-back operation $f^* : \mathcal{D}'_\Gamma(\Omega_2) \to \mathcal{D}'_{f^*\Gamma}(\Omega_1)$ is continuous for the normal topology.

We will show this by proving that $\langle f^*u, v\rangle$ is continuous for every v in an equicontinuous set. Before doing so, we characterize the equicontinuous sets of the normal topology, which is of independent interest.

5.1 Equicontinuous subsets

Let Ω be open in $\mathbb{R}^d$ and Γ be a closed cone in $\dot{T}^*\Omega$. We define the open cone $\Lambda = \{(x, \xi) \in \dot{T}^*\Omega\,; (x, -\xi) \notin \Gamma\}$ and the space $\mathcal{E}'_\Lambda(\Omega)$ of compactly supported distributions $v \in \mathcal{E}'(\Omega)$ such that $\mathrm{WF}(v) \subset \Lambda$.

Acccording to Definition (3.2), a set H is equicontinuous in $\mathcal{E}'_\Lambda(\Omega)$ (which is the strong dual of $\mathcal{D}'_\Gamma(\Omega)$ [5]) if and only if there is a finite number of seminorms $||\cdot||_{N_1,V_1,\chi_1},\dots,||\cdot||_{N_k,V_k,\chi_k}$ of $\mathcal{D}'_\Gamma(\Omega)$, a bounded subset B_0 of $\mathcal{D}(\Omega)$ and a constant M such that

$$(5.1) \qquad |\langle u,v\rangle| \;\le\; M\sup\{||u||_{N_1,V_1,\chi_1},\dots,||u||_{N_k,V_k,\chi_k},p_{B_0}(u)\}$$

for every $u\in\mathcal{D}'_\Gamma(\Omega)$ and every $v\in H$. There is only one seminorm p_{B_0} because these seminorms are saturated [13, p. 107] in $\mathcal{D}'(\Omega)$ with the strong topology. The following theorem will be useful to prove the continuity of linear maps [13, p. 200]:

Theorem 5.2. *If E is a locally convex space and $f:E\to\mathcal{D}'_\Gamma(\Omega)$ is a linear map, then f is continuous if and only if, for every equicontinuous set H of $\mathcal{E}'_\Lambda(\Omega)$ the seminorm $p_H:E\to\mathbb{R}$ defined by $p_H(x)=\sup_{v\in H}|\langle f(x),v\rangle|$ is continuous.*

The equicontinuous sets of $\mathcal{E}'_\Lambda(\Omega)$ are known:

Lemma 5.3. *A subset B of $\mathcal{E}'_\Lambda(\Omega)$ is equicontinuous if and only if there is: (i) a compact set $K\subset\Omega$ containing the support of all elements of B; (ii) a closed cone $\Xi\subset\Lambda$ such that $B\subset\mathcal{D}'_\Xi(\Omega)$, B is bounded in $\mathcal{D}'_\Xi(\Omega)$ and $\underline{\pi}(\Xi)\subset K$.*

Proof. We first prove that every such B is equicontinuous. We showed in [5] that the space $\mathcal{E}'_\Lambda(\Omega)$ is the inductive limit of spaces $E_\ell=\{v\in\mathcal{E}'_\Lambda(\Omega)\,;\operatorname{supp}v\in L_\ell,\mathrm{WF}(v)\in\Lambda_\ell\}$, where the compact sets L_ℓ exhaust Ω and the closed cones Λ_ℓ exhaust Λ. Thus, there is an integer ℓ such that $\Xi\subset\Lambda_\ell$ and $B\subset E_\ell$. The inclusion of Ξ in Λ_ℓ implies that every seminorm $||\cdot||_{N,V,\chi}$ of E_ℓ is also a seminorm of $\mathcal{D}'_\Xi(\Omega)$ because $\operatorname{supp}\chi\times V$ does not meet Ξ if it does not meet Λ_ℓ. Thus, B is bounded in E_ℓ and Eq. (8) of [5] gives us

$$\sup_{v\in B}|\langle u,v\rangle| \;\le\; \sum_j\Big(p_{B_j}(u)+||u||_{m+n+1,V_j,\chi_j}CI_n^{n+1}+||u||_{n,V_j,\chi_j}M_{n,W_j,\chi_j}I_n^{2n}\Big),$$

which can be converted to the equicontinuity condition (5.1).

To show the converse, we denote by B the set of all $v\in\mathcal{E}'_\Lambda(\Omega)$ that satisfy Eq. (5.1). Then, by following exactly the proof of Prop. 7 of [5], we obtain that the support of all elements of B is included in a compact set $K=\cup_j\operatorname{supp}\chi_j\cup K'$, where K' is a compact set containing the support of all $f\in B_0$. Moreover, the wave front set of all elements of B is contained in $\Xi=\cup_j\operatorname{supp}\chi_j\times(-V_j)$. It remains to show that B is bounded in $\mathcal{D}'_\Xi(\Omega)$ for the normal topology. We first notice that, if $\operatorname{supp}\chi\times(-V)\subset\Xi$, then $||\cdot||_{N,V,\chi}$ is a continuous seminorm of the strong dual $\mathcal{E}'_{(\Xi')^c}(\Omega)$ of $\mathcal{D}'_{\Xi}(\Omega)$. Indeed, it was shown in the proof of Prop. 7 of [5] that $||u||_{N,V,\chi} = \sup_{\xi\in V} |\langle u, f_\xi\rangle|$, where $f_\xi(x) = (1+|\xi|)^N\chi(x)e^{-i\xi\cdot x}$ and the set $\{f_\xi, \xi \in V\}$ is bounded in $\mathcal{D}'_{\Xi}(\Omega)$. If B' is a bounded set in $\mathcal{E}'_{(\Xi')^c}(\Omega)$, the continuous seminorms $||u||_{N,V,\chi}$ and $p_{B_0}(u)$ of $\mathcal{E}'_{(\Xi')^c}(\Omega)$ appearing on the right hand side of ((5.1)) are bounded over B'. Thus, for any bounded set B' in $\mathcal{E}'_{(\Xi')^c}(\Omega)$, taking $u \in B'$ and taking the sup in ((5.1)) over $u \in B'$ yields that $\sup_{u\in B', v\in B} |\langle u, v\rangle|$ is bounded and B is a bounded subset of $\mathcal{D}'_{\Xi}(\Omega)$ when $\mathcal{D}'_{\Xi}(\Omega)$ is equipped with the strong $\beta(\mathcal{D}'_{\Xi}, \mathcal{E}'_{(\Xi')^c})$ topology. It is shown in [5, Theorem 33] that the bounded sets of $\mathcal{D}'_{\Gamma}(\Omega)$ coincide for the strong and the normal topologies. Thus, B is bounded for the normal topology. □

We obtain the following characterization of continuous linear maps:

Theorem 5.4. *Let E be a locally convex space, Ω an open subset of $\mathbb{R}^d$ and Γ a closed cone in $\dot{T}^*\Omega$. A linear map $f : E \to \mathcal{D}'_{\Gamma}(\Omega)$ is continuous if and only if every map $f_B : E \to \mathbb{R}$ defined by $f_B(x) = \sup_{v\in B} |\langle f(x), v\rangle|$ is continuous, where B is equicontinuous in $\mathcal{E}'_{\Lambda}(\Omega)$, with $\Lambda = (\Gamma')^c$.*

The equicontinuous sets of $\mathcal{E}'_{\Lambda}(\Omega)$ intervene also because the duality pairing enjoys a sort of hypocontinuity where, for $\mathcal{E}'_{\Lambda}(\Omega)$, the bounded sets are replaced by the equicontinuous ones:

Theorem 5.5. *Let the duality pairing $\mathcal{D}'_{\Gamma}(\Omega) \times \mathcal{E}'_{\Lambda}(\Omega) \to \mathbb{K}$ be defined by $u \times v \to f(u, v) = \langle u, v\rangle$. Then, for every bounded set A of $\mathcal{D}'_{\Gamma}(\Omega)$ and every equicontinuous set B of $\mathcal{E}'_{\Lambda}(\Omega)$ the sets of maps $\{f_u\,; u \in A\}$ and $\{f_v\,; v \in B\}$ are equicontinuous [14, p. 157].*

5.2 Proof of continuity of the pull-back

Strategy of the proof. Let Ω_1 and Ω_2 be open sets in $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$, respectively. Let $f : \Omega_1 \to \Omega_2$ be a smooth map and Γ be a closed cone in $\dot{T}^*\Omega_2$. We want to show that the pull-back $f^* : \mathcal{D}'_{\Gamma}(\Omega_2) \to \mathcal{D}'_{f^*\Gamma}(\Omega_1)$ is continuous for the normal topology. According to Theorem 5.4, the pull-back is continuous if and only if, for every equicontinuous set $B \subset \mathcal{E}'_{\Lambda}(\Omega_1)$ (where $\Lambda = (f^*\Gamma)^{',c}$) the family of maps $(\rho_v)_{v\in B}$, defined by $\rho_v : u \mapsto \langle f^*u, v\rangle$, is equicontinuous which implies that $\sup_{v\in B} |\,\langle f^*u, v\rangle\,|$ is continuous in u. By Lemma 5.3, we know that there is a compact set $K \subset \Omega_1$ and a closed cone $\Xi \subset (f^*\Gamma')^c$ such that $\operatorname{supp} v \subset K$ and $\mathrm{WF}(v) \subset \Xi$ for all $v \in B$. Choose a function $\chi \in \mathcal{D}(\Omega_1)$ such that $\chi|_K = 1$. If $(\varphi_i)_{i\in I}$ is a partition of unity of Ω_2, we can write $\langle f^*u, v\rangle = \sum_i \langle f^*(u\varphi_i), v\chi\rangle$. The image of $\operatorname{supp}\chi$ by f being compact [2, p. 19], only a finite number of terms of this sum are nonzero and the family ρ_v is equicontinuous if and only if, for every $\varphi \in \mathcal{D}(\Omega_2)$, the family of maps $u \mapsto \langle f^*(u\varphi), v\chi\rangle$ is equicontinuous.

Stationary phase and Schwartz kernels. In order to calculate the pairing between $f^*(u\varphi)$ and v, we first notice that, when u is a locally integrable function, then $u\varphi(y) = \mathcal{F}^{-1}(\widehat{u\varphi})(y) = (2\pi)^{-d_2}\int_{\mathbb{R}^{d_2}} d\eta e^{i\eta\cdot y}\widehat{u\varphi}(\eta)$, so that $f^*\left(u\varphi\right)(x) = (2\pi)^{-d_2}\int_{\mathbb{R}^{d_2}} d\eta e^{i\eta\cdot f(x)}\widehat{u\varphi}(\eta)$ and

$$\begin{aligned}\langle f^*\left(u\varphi\right), \chi v\rangle &= \frac{1}{(2\pi)^{d_2}}\int_{\Omega_1}\int_{\mathbb{R}^{d_2}} \chi(x)v(x)e^{i\eta\cdot f(x)}\widehat{u\varphi}(\eta)dxd\eta \\ &= \frac{1}{(2\pi)^{d_2}}\int_{\mathbb{R}^{d_2}}\int_{\Omega_1}\int_{\mathbb{R}^{d_2}} \chi(x)v(x)e^{i\eta\cdot f(x)}e^{-iy\cdot\eta}u(y)\varphi(y)dydxd\eta.\end{aligned}$$

This definition can be extended to any distribution $u \in \mathcal{D}'_\Gamma$ as

$$\langle f^*(u\varphi), \chi v\rangle = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} u(y)v(x)I(x,y)dxdy, \tag{5.2}$$

where $d = d_1 + d_2$, $I(x,y) = \int_{\mathbb{R}^{d_2}} e^{i\eta\cdot(f(x)-y)}\varphi(y)\chi(x)d\eta$. The duality pairing can also be written $\langle f^*(u\varphi), v\chi\rangle = \langle v\otimes u, I\rangle$. Note that $I(x,y) = (2\pi)^{-d_2}\chi(x)\varphi(y)\int d\eta e^{i\eta\cdot(f(x)-y)}$ is an oscillatory integral [12] with symbol $\chi(x)\varphi(y)$ and phase $\eta\cdot(f(x)-y)$ where $\eta\cdot(f(x)-y)$ is homogeneous of degree 1 with respect to η, for all $\eta \neq 0, d\left(\eta\cdot(f(x)-y)\right) \neq 0$. Therefore, $I \in \mathcal{D}'(\Omega_1\times\Omega_2)$ is the Schwartz kernel of the bilinear continuous map: $(u,v) \mapsto \langle f^*(u\varphi), v\chi\rangle$.

Proof of Proposition 5.1. By Theorem 4.6, the map $(v,u) \mapsto v\otimes u$ is hypocontinuous from $\mathcal{D}'_\Xi\times\mathcal{D}'_\Gamma$ to $\mathcal{D}'_{\Gamma_\otimes}$ where $\Gamma_\otimes = \Xi\times\Gamma\cup(\Omega_1\times\{0\})\times\Gamma\cup\Xi\times(\Omega_2\times\{0\})$. Let $\Lambda_\otimes$ be the open cone $\Gamma'^{;c}_\otimes$. Therefore by Theorem 5.5, the family of duality pairings $u\otimes v \in \mathcal{D}'_{\Gamma_\otimes} \mapsto \langle u\otimes v, r\rangle$ is equicontinuous from $\mathcal{D}'_{\Gamma_\otimes}$ to $\mathbb{K}$ uniformly in $r \in B'$ for every equicontinuous set B' of $\mathcal{E}'_{\Lambda_\otimes}$. In particular, if B' contains only the element I, then the map $v\otimes u \mapsto \langle v\otimes u, I\rangle$ would be continuous since I is compactly supported in $\operatorname{supp}\chi\times\operatorname{supp}\varphi$ and its wave front set is contained in $\Lambda_\otimes$. Thus, if $\mathrm{WF}(I) \subset \Lambda_\otimes$, then the map $(v,u) \mapsto \langle v\otimes u, I\rangle$ is hypocontinuous by the next lemma.

Lemma 5.6. *The composition of a hypocontinuous map by a continuous linear map is hypocontinuous.*

Proof. Let $f : E\times F \to G$ be a hypocontinuous map and $g : G \to H$ a continuous linear map. The map $g\circ f$ is hypocontinuous if and only if, for every bounded set $B \subset F$ and every neighborhood W of zero in H, there is a neighborhood U of zero in E such that $(g \circ f)(U \times B) \subset W$ (with the similar condition for $(g \circ f)(A \times V)$). By the continuity of g, there is a neighborhood Z of zero in G such that $g(Z) \subset W$. By the hypocontinuity of f, there is a neighborhood U of zero in E such that $f(U \times B) \subset Z$. Thus, $(g \circ f)(U \times B) \subset g(Z) \subset W$. $\square$

Therefore, the map $(v, u) \mapsto \langle f^*(u\varphi), \chi v\rangle$ is hypocontinuous, by item (i) of Definition 3.1, this implies that the family of maps $\rho_v : u \mapsto \langle f^*(u\varphi), \chi v\rangle$ with $v \in B$ is equicontinuous. It just remains to check that $\mathrm{WF}(I) \subset \Lambda_\otimes$, i.e. that $\mathrm{WF}(I)'$ does not meet $\Gamma_\otimes$. The wave front set of I is $\mathrm{WF}(I) \subset \{(x, f(x); -\eta \circ d_x f, \eta)\,; x \in \operatorname{supp} \chi\}$ [12, p. 260]. Recall that $\Xi \subset (f^*\Gamma')^c = \{(x, -\eta \circ df_x)\,; (f(x), \eta) \notin \Gamma\}$. By definition of $\Gamma_\otimes$ we must satisfy the following three conditions:

- $\Xi \times \Gamma \cap \mathrm{WF}(I)' = \emptyset$ because it is the set of points $(x, f(x); -\eta \circ d_x f, \eta)$ such that $(f(x), \eta) \notin \Gamma$ by definition of Ξ and $(f(x), \eta) \in \Gamma$ by definition of Γ;
- $\Xi \times (\Omega_2 \times \{0\}) \cap \mathrm{WF}(I)' = \emptyset$ because we would need $\eta = 0$ whereas $(y, \eta) \in \Gamma$ implies $\eta \neq 0$;
- $(\operatorname{supp} \chi \times \{0\}) \times \Gamma \cap \mathrm{WF}(I)' \subset \{(x, f(x); 0, \eta)\,; x \in \operatorname{supp} \chi, \eta \circ df_x = 0, (f(x), \eta) \in \Gamma\}$.

Thus, if $f^*\Gamma \cap N_f = \emptyset$, then $\mathrm{WF}(I)' \cap \Gamma_\otimes = \emptyset$ and the pull-back is continuous.

How to write the pull–back operator in terms of the Schwartz kernel I ? Relationship with the product of distributions. We start from a linear operator $L : \mathcal{D}(\mathbb{R}^{d_2}) \longmapsto \mathcal{D}'(\mathbb{R}^{d_1})$ with corresponding Schwartz kernel $K \in \mathcal{D}'(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Using the standard operations on distributions, we can make sense of the well–known representation formula $Lu = \int_{\mathbb{R}^{d_2}} K(x, y)u(y)dy$ for an operator $L : \mathcal{D}(\mathbb{R}^{d_2}) \longmapsto \mathcal{D}'(\mathbb{R}^{d_1})$ and its kernel $K \in \mathcal{D}'(\mathbb{R}^{d_1} \times \mathbb{R}^{d_2})$. Let us define the two projections $\pi_2 := (x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \longmapsto y \in \mathbb{R}^{d_2}$ and $\pi_1 := (x, y) \in \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \longmapsto x \in \mathbb{R}^{d_1}$, then we define $K(x, y)u(y) = K(x, y)\pi_2^* u(x, y) = K(x, y)\,(1(x) \otimes u(y))$ where $\pi_2^* u = 1 \otimes u$ and $\int_{\mathbb{R}^{d_2}} dy f(x, y) = \pi_{1*} f(x)$. Therefore

$$Lu = \int_{\mathbb{R}^{d_2}} K(x, y)u(y)dy = \pi_{1*}\left(K\left(\pi_2^* u\right)\right). \tag{5.3}$$

The interest of the formula $Lu = \pi_{1*}\left(K\left(\pi_2^* u\right)\right)$ is that everything generalizes to oriented manifolds. Replace $\mathbb{R}^{d_2}$ (resp $\mathbb{R}^{d_1}$) with a manifold

M_2 (resp M_1) with smooth volume densities $|\omega_2|$ (resp $|\omega_1|$), the duality pairing is defined as the extension of the usual integration against the volume densities, for instance: $\forall(u,\varphi) \in C^\infty(M_1)\times\mathcal{D}(M_1), \langle u,\varphi\rangle_{M_1} = \int_{M_1}(u\varphi)\omega_1$. Finally, for the linear continuous map $L := u \in \mathcal{D}(\mathbb{R}^{d_2}) \longmapsto \chi f^*(u\varphi) \in \mathcal{D}'(\mathbb{R}^{d_1})$, we get the formula: $Lu = \pi_{1*}\left(I(\pi_2^* u)\right)$ where $I(x,y) = (2\pi)^{-d_2}\chi(x)\varphi(y)\int d\eta e^{i\eta\cdot(f(x)-y)}$ is the Schwartz kernel of L.

5.3 Pull-back by families of smooth maps

To renormalize quantum field theory in curved spacetimes, it will be crucial to pull-back by family of smooth maps. We start with a simple lemma.

Lemma 5.7. *Let Ω_1,Ω_2,U be open sets in $\mathbb{R}^{d_1},\mathbb{R}^{d_2},\mathbb{R}^n$ respectively. For any compact sets $(K_1\subset\Omega_1, K_2\subset\Omega_2, A\subset U)$ and f a smooth map $f:\Omega_1\times U\to\Omega_2$, the conic set*

$$\Gamma = \{(x,f(x,a);-\eta\circ d_xf(x,a),\eta)\,;(x,a,f(x,a))\in K_1\times A\times K_2,\eta\neq 0\}$$

is closed in $\dot{T}^*\left(\Omega_1\times\Omega_2\right)$.

Proof. Let $(x,y;\xi,\eta)\in\overline{\Gamma}$ such that $(\xi,\eta)\neq(0,0)$. Then there is a sequence

$$(x_n,f(x_n,a_n);-\eta_n\circ d_xf(x_n,a_n),\eta_n)\in\Gamma,(x_n,a_n,f(x_n,a_n))\in K_1\times A\times K_2$$

which converges to $(x,y;\xi,\eta)$. By compactness of A, we extract a convergent subsequence $a_n\to a$. By continuity of d_xf, we find that $\xi=-\eta\circ d_xf(x,a)$, we also find that $\lim_{n\to\infty} f(x_n,a_n) = f(x,a)\in K_2$ since K_2 is closed and we finally note that we must have $\eta\neq 0$ otherwise $\xi=0,\eta=0$. Therefore $(x,y;\xi,\eta)\in\Gamma$ by definition. Finally, $\overline{\Gamma}\subset\Gamma$ hence Γ is closed. □

Proposition 5.8. *Let Ω_1 be an open set in $\mathbb{R}^{d_1}$, $A\subset U\subset\mathbb{R}^n$ where A is compact, U and Ω_2 are open sets in $\mathbb{R}^{d_2}$. Let $\chi\in\mathcal{D}(\Omega_1)$, $\varphi\in\mathcal{D}(\Omega_2)$ and f a smooth map $f:\Omega_1\times U\to\Omega_2$.*

1. *Then the family of distributions $(I_{f(.,a)})_{a\in A}$ formally defined by*

$$I_{f(.,a)}(x,y) \;=\; \chi(x)\varphi(y)\int_{\mathbb{R}^{d_2}}\frac{d\theta}{(2\pi)^{d_2}}e^{i\theta\cdot(f(x,a)-y)}$$

is a bounded set in $\mathcal{D}'_\Gamma$, where Γ is the closed cone in $\dot{T}^*(\Omega_1\times\Omega_2)$ defined by:

$$\begin{aligned}\Gamma \;=\; &\{(x,f(x,a);-\eta\circ d_xf(x,a),\eta)\,;\\ &x\in\operatorname{supp}\chi, f(x,a)\in\operatorname{supp}\varphi, a\in A,\eta\neq 0\}.\end{aligned}$$

2. *For any open cone Λ containing Γ, $(I_{f(.,a)})_{a\in A}$ is equicontinuous in $\mathcal{E}'_\Lambda(\Omega_1\times\Omega_2)$.*

We will use the pushforward Theorem 6.3, whose proof will be given in Section 7, in the following proof.

Proof. From Lemma 5.3 and from the fact that $(I_{f(.,a)})_{a\in A}$ is supported in a fixed compact set $\operatorname{supp}\chi\times\operatorname{supp}\varphi$, we deduce that conclusion (2) follows from the first claim thus it suffices to prove the claim (1).

The conic set Γ is closed by Lemma 5.7. To prove that the family $\left(I_{f(.,a)}\right)_{a\in A}$ is bounded in $\mathcal{D}'_\Gamma$, it suffices to check that $\forall v\in\mathcal{E}'_{\Gamma'^c}(\Omega_1\times\Omega_2)$, $\sup_{a\in A}|\left\langle I_{f(.,a)},v\right\rangle|<+\infty$ because of [5, Proposition 1].

Step 1 Our goal is to study the map $a\longmapsto\int_{\Omega_1\times\Omega_2}I_f(x,y,a)v(x,y)$ where

$$I_f(x,y,a)=\chi(x)\varphi(y)\int_{\mathbb{R}^{d_2}}\frac{d\theta}{(2\pi)^{d_2}}e^{i\theta\cdot(f(x,a)-y)}$$

Let π_{12},π_3 be projections from $\Omega_1\times\Omega_2\times U$ defined by $\pi_{12}(x,y,a)=(x,y)$ and $\pi_3(x,y,a)=a$. Using the dictionary explained in paragraph 5.2, if v were a test function, then we would find that

$$\int_{\Omega_1\times\Omega_2}I_f(x,y,.)v(x,y)dxdy \;=\; \pi_{3*}\left(I_f\pi_{12}^*v\right)\in\mathcal{D}'(U). \tag{5.4}$$

We want to prove that $a\longmapsto\int_{\Omega_1\times\Omega_2}I_f(x,y,a)v(x,y)dxdy$ is smooth in some open neighborhood of A since this would imply that

$$\sup_{a\in A}|\int_{\Omega_1\times\Omega_2}I_f(x,y,a)v(x,y)dxdy|=\sup_{a\in A}|\left\langle I_{f(.,a)},v\right\rangle|<+\infty.$$

In order to do so, it suffices to prove that the condition $v\in\mathcal{E}'_{\Gamma',c}$ implies that the distributional product $I_f(x,y,a)v(x,y)=I_f\left(\pi_{12}^*v\right)(x,y,a)$ makes sense in $\mathcal{D}'(\Omega_1\times\Omega_2\times U)$ and the push–forward $\pi_{3*}\left(I_f\pi_{12}^*v\right)=\int_{\Omega_1\times\Omega_2}I_f(x,y,.)v(x,y)dxdy$ has empty wave front set over some open neighborhood of A.

Step 2 The wave front set $WF(I_f)$ is the set of all

$$\begin{pmatrix} x & ; & -\theta\circ d_xf\\ f(x,a) & ; & \theta\\ a & ; & -\theta\circ d_af\end{pmatrix}$$

such that $x\in\operatorname{supp}\varphi, f(x,a)\in\operatorname{supp}\chi, a\in U,\theta\neq0$. And the wave front set $WF(\pi_{12}^*v)$ is the set of all $\begin{pmatrix} x & ; & \xi\\ y & ; & \eta\\ a & ; & 0\end{pmatrix}$ such that $\begin{pmatrix} x & ; & \xi\\ y & ; & \eta\end{pmatrix}\in WF(v)$.

One also have $v \in \mathcal{E}'_{\Gamma'^c}$ implies $WF(v) \cap \Gamma' = \emptyset$ so that $\xi \neq -\eta \circ d_x f$ and

$$\forall \theta, \begin{pmatrix} \xi - \theta \circ d_x f \\ \theta + \eta \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \text{has no solution}$$

Observe that

$$WF(I_f) + WF(\pi_{12}^* v) = \{ \begin{pmatrix} x & ; & \xi - \theta \circ d_x f \\ f(x) & ; & \theta + \eta \\ a & ; & -\theta \circ d_a f \end{pmatrix}, \forall \theta \in \mathbb{R}^d \setminus \{0\}\},$$

implies $(WF(I_f) + WF(\pi_{12}^* v)) \cap \underline{0} = \emptyset$ and $(WF(I_f) \cup WF(\pi_{12}^* v)) \cap \underline{0} = \emptyset$.

Step 3 In the last step, we shall prove that the condition $WF(v) \cap \Gamma' = \emptyset$ actually implies that $WF(\pi_{3*}(I_f \pi_{12}^* v))$ is empty over some open neighborhood U' of A. The condition $WF(v) \cap \Gamma' = \emptyset$ implies the existence of some open neighborhood U' of A s.t.

$$\forall a \in U', \forall (x, f(x,a); \xi, \eta) \in WF(v), \xi \neq -\eta \circ d_x f(x,a).$$

Since A and supp v are compact and $A \times WF(v)$ is closed, we can find $\delta > 0$ s.t.

$$\forall (a, (x, f(x,a); \xi, \eta)) \in A \times WF(v), \ \ |\xi + \eta \circ d_x f(x,a)| \geqslant \delta |\eta|.$$

Define $U' = \{a \in U\, ; \forall (x, f(x,a); \xi, \eta) \in WF(v), |\xi + \eta \circ d_x f(x,a)| > \frac{\delta}{2}|\eta|\}$. Therefore, the condition $WF(v) \cap \Gamma' = \emptyset$ on the wave front set of v ensures that $\pi_{3*}(I_f \pi_{12}^* v)$ is well defined in $\mathcal{D}'_{\emptyset}(U') = C^\infty(U')$. Hence $a \longmapsto \langle v, I_{f(.,a)} \rangle = \int_{\Omega_1 \times \Omega_2} I_f(x,y,a) v(x,y) dx dy$ is smooth on U', a fortiori continuous on the compact set A which means that $\sup_{a \in A} | \langle v, I_{f(.,a)} \rangle | < +\infty$. □

Theorem 5.9. *Let $\Omega_1 \subset \mathbb{R}^{d_1}, \Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets, $A \subset U \subset \mathbb{R}^n$ where A compact, U open and Γ a closed cone in $\dot{T}^*\Omega_2$. Let $f : \Omega_1 \times U \to \Omega_2$ be a smooth map such that $\forall a \in A, f(.,a)^*\Gamma$ does not meet the zero section $\underline{0}$ and set $\Theta = \bigcup_{a \in A} f(.,a)^*\Gamma$. Then for all seminorms P_B of $\mathcal{D}'_\Theta(\Omega_1)$, $\forall u \in \mathcal{D}'_\Gamma(\Omega_2)$, the family $P_B(f(.,a)^* u)_{a \in A}$ is bounded.*

Proof. We need to prove that $\sup_{(v,a) \in B \times A} |\langle f(.,a)^* u, v \rangle| < +\infty$ for any equicontinuous subset B of $\mathcal{E}'_{\Theta',c}(\Omega_1)$. It follows from Lemma 5.3 that there exists some closed cone Ξ such that $\Xi \cap \Theta' = \emptyset$ and $B \subset \mathcal{D}'_\Xi(\Omega_1)$. Set $\Gamma_\otimes = (\Xi \times \Gamma) \cup ((\Omega_1 \times \{0\}) \times \Gamma) \cup (\Xi \times (\Omega_2 \times \{0\}))$. Let $\Lambda_\otimes$ be the open cone defined as $\Lambda_\otimes = (\Gamma_\otimes)^{\prime,c}$. By proposition 5.8, we can easily verify as in the proof of Proposition 5.1 that the family $(I_{f(.,a)})_{a \in A}$ is equicontinuous in $\mathcal{E}'_{\Lambda_\otimes}$. Then we prove similarly as in the proof of the pull-back Proposition

5.1 that the family of maps $\rho_{v,a} : u \mapsto \langle f(.,a)^*(u\varphi), \chi v\rangle$ with $(v,a) \in B \times A$ is equicontinuous where B is equicontinuous in $\mathcal{E}'_{\Theta',c}(\Omega_1)$ and χ is chosen in such a way that $\chi|_{\text{supp } B} = 1$ and $\varphi|_{f(\text{supp } B)} = 1$, therefore:

$$\sup_{(v,a)\in B\times A} |\langle f(.,a)^*u, v\rangle| = \sup_{(v,a)\in B\times A} |\langle f(.,a)^*(u\varphi), \chi v\rangle| < +\infty.$$

and the result follows from Theorem 5.4. □

6 Product and push–forward of distributions

Hörmander noticed that the product of distributions u and v can be described as the composition of the tensor product $(u,v) \mapsto u \otimes v$ with the pull-back by the map $f : x \mapsto (x,x)$. If the wave front sets of u and v are contained in Γ_1 and Γ_2, then the wave front set of $u \otimes v$ is contained in $\Gamma_\otimes = (\Gamma_1 \times \Gamma_2) \cup ((\Omega_1 \times \{0\}) \times \Gamma_2) \cup (\Gamma_1 \times (\Omega_2 \times \{0\}))$ and the pull-back is well-defined if the set $N_f = \{(x,x;\eta_1,\eta_2)\,;(\eta_1+\eta_2)\circ dx) = 0\}$, which is the conormal bundle of the diagonal $\Delta \subset \mathbb{R}^n \times \mathbb{R}^n$, does not meet Γ, i.e. if there is no point $(x;\eta)$ in Γ_1 such that $(x;-\eta)$ is in Γ_2. This gives us

Theorem 6.1. *Let $\Omega \subset \mathbb{R}^n$ be an open set and Γ_1, Γ_2 be two closed cones in $\dot{T}^*\Omega$ such that $\Gamma_1 \cap \Gamma_2' = \emptyset$. Then the product of distributions is hypocontinuous for the normal topology from $\mathcal{D}'_{\Gamma_1} \times \mathcal{D}'_{\Gamma_2}$ to $\mathcal{D}'_{\Gamma}$, where*

$$(6.1) \quad \Gamma \;=\; \big(\Gamma_1 \times_\Omega \Gamma_2\big) \cup \big((\Omega \times \{0\}) \times_\Omega \Gamma_2\big) \cup \big(\Gamma_1 \times_\Omega (\Omega \times \{0\})\big).$$

Proof. The product of distribution is the composition of the hypocontinuous tensor product with the continuous pull-back (see Lemma 5.6). □

This theorem has the useful corollary:

Corollary 6.2. *Let $\Omega \subset \mathbb{R}^n$ be an open set and Γ be a closed cone in $\dot{T}^*\Omega$. Then the product of a smooth map and a distribution is hypocontinuous for the normal topology from $C^\infty(\Omega) \times \mathcal{D}'_\Gamma$ to $\mathcal{D}'_\Gamma$.*

Proof. By Lemma 7.2, $C^\infty(\Omega)$ and $\mathcal{D}'_\emptyset$ are topologically isomorphic. Therefore, the corollary follows by applying Theorem 6.1 to $\Gamma_1 = \emptyset$ and $\Gamma_2 = \Gamma$. Equation (6.1) shows that the wave front set of the product is in Γ. □

6.1 The push–forward as a consequence of the pull–back theorem.

Theorem 6.3. *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets and Γ a closed cone in $\dot{T}^*\Omega_1$. For any smooth map $f : \Omega_1 \to \Omega_2$ and any closed subset C of*

Ω_1 such that $f|_C : C \to \Omega_2$ is proper and $\underline{\pi}(\Gamma) \subset C$, then f_* is continuous in the normal topology from $\{u \in \mathcal{D}'_\Gamma\,; \operatorname{supp} u \subset C\}$ to $\mathcal{D}'_{f_*\Gamma}$, where $f_*\Gamma = \{(y;\eta) \in \dot{T}^*\Omega_2\,; \exists x \in \Omega_1$ with $y = f(x)$ and $(x;\eta\circ df_x) \in \Gamma \cup \operatorname{supp} u \times \{0\}\}$.

Proof. The idea of the proof is to think of a push–forward as the adjoint of a pull–back [8]. For all B equicontinuous in $\mathcal{E}'_\Lambda$ where $\Lambda = f_*\Gamma'^{,c}$, for all $v \in B, \operatorname{supp}\,(f^*v) \cap C$ is contained in a fixed compact set K since f is proper on C. Hence, for any $\chi \in \mathcal{D}(\mathbb{R}^d)$ such that $\chi|_C = 1$ and f is proper on supp χ, we should have at least formally $\langle f_*u, v\rangle = \langle u, \chi f^*v\rangle$ if the duality pairings make sense. On the one hand, if $v \in \mathcal{E}'_\Lambda(\Omega_2)$ where $f^*\Lambda$ does not meet the zero section $\underline{0} \subset T^*\Omega_1$ then the pull–back f^*v would be well defined by the pull–back Proposition (5.1) (which is equivalent to the fact that $N_f \cap \Lambda = \emptyset$). On the other hand, the duality pairing $\langle u, \chi(f^*v)\rangle$ is well defined if $f^*\Lambda \cap \Gamma' = \emptyset$. Combining both conditions leads to the requirement that $f^*\Lambda \cap (\Gamma' \cup \underline{0}) = \emptyset$. But note that:

$$\begin{aligned}
& (f^*\Lambda) \cap (\Gamma' \cup \underline{0}) = \emptyset \\
\Leftrightarrow\ & \{(x;\eta\circ df) | (f(x);\eta) \in \Lambda, (x;\eta\circ df) \in \Gamma' \cup \{0\}\} = \emptyset \\
\Leftrightarrow\ & (f(x);\eta) \in \Lambda \implies (x;\eta\circ df) \notin \Gamma' \cup \{0\} \\
\Leftrightarrow\ & (f(x);\eta) \in \Lambda' \implies (x;\eta\circ df) \notin \Gamma \cup \{0\}.
\end{aligned}$$

which is equivalent to the fact that Λ' does not meet $f_*\Gamma = \{(f(x);\eta)\,; (x;\eta\circ df) \in (\Gamma \cup \underline{0})\,, \eta \neq 0\} \subset T^*\Omega_2$ which is exactly the assumption of our theorem. Therefore, the set of distributions χf^*B is supported in a fixed compact set, bounded in $\mathcal{D}'_{f^*\Lambda}(\Omega_1)$ by the pull–back Proposition 5.1 applied to f^*, the duality pairings are well defined and: $\sup_{v\in B} |\langle f_*u, v\rangle| = \sup_{v\in B'} |\langle u, v\rangle|$ where $B' = \chi f^*B$ is equicontinuous in $\mathcal{E}'_{\Gamma'^{,c}}(\Omega_1)$ (the support of the distribution is compact because f is proper) which means that $\sup_{v\in B'} |\langle u, v\rangle|$ is a continuous seminorm for the normal topology of $\mathcal{D}'_\Gamma(\Omega_1)$. □

We state and prove a parameter version of the push–forward theorem

Theorem 6.4. *Let $\Omega_1 \subset \mathbb{R}^{d_1}$ and $\Omega_2 \subset \mathbb{R}^{d_2}$ be two open sets, $A \subset U \subset \mathbb{R}^n$ where A is compact, U is open and Γ a closed cone in $\dot{T}^*\Omega_1$. For any smooth map $f : \Omega_1 \times U \to \Omega_2$ and any closed subset C of Ω_1 such that $f : C \times A \to \Omega_2$ is proper and $\underline{\pi}(\Gamma) \subset C$, then $f(.,a)_*$ is uniformly continuous in the normal topology from $\{u \in \mathcal{D}'_\Gamma\,; \operatorname{supp} u \subset C\}$ to $\mathcal{D}'_\Xi$, where $\Xi = \cup_{a\in A} f(.,a)_*\Gamma$.*

Proof. We have to check that Ξ is closed over $\dot{T}^*_{f(C\times A)}\Omega_2$. Let $(y;\eta) \in \overline{\Xi} \cap \dot{T}^*_{f(C\times A)}\Omega_2$ then there exists a sequence $(y_n;\eta_n) \to (y;\eta)$ such that $(y_n;\eta_n) \in \Xi \cap \dot{T}^*_{f(C\times A)}\Omega_2$. By definition, $y_n = f(x_n, a_n)$ where $(x_n, \eta_n \circ d_x f(x_n, a_n)) \in$

$\Gamma \cup \underline{0}, (x_n, a_n) \in C \times A$. The central observation is that $\overline{\{y_n | n \in \mathbb{N}\}} \subset f(C \times A)$ and $\overline{\{(x_n, a_n) | f(x_n, a_n) = y_n, n \in \mathbb{N}\}} \subset C \times A$ are compact sets because f is proper on $C \times A$. Then we can extract convergent subsequences $(x_n, a_n) \to (x, a)$ and $(x, \eta \circ d_x f(x, a)) \in \Gamma \cup \underline{0}$ since $\Gamma \cup \underline{0}$ is closed in $\dot{T}^*\Omega_2$ and $d_x f$ is continuous. By definition of Ξ, this proves that $(y; \eta) \in \Xi$ and we can conclude that Ξ is closed. Now we can repeat the proof of the push–forward theorem except that we use the pull–back theorem with parameters. Let B be equicontinuous in $\mathcal{E}'_{\Xi',c}(\Omega_2)$ hence all elements of B have support contained in some compact. We have $\forall v \in B, \text{supp } (f^*v) \cap C \times A$ is compact, therefore for $\chi = 1$ on $\cup_{a \in A} \text{supp } (f(., a)^*v) \cap C$, the family $B' = \{(\chi f(., a)^* v) | a \in A, v \in B\}$ is equicontinuous in $\mathcal{E}'_{\Theta}$ where $\Theta = \Gamma'^{,c}$ by the parameter version of the pull–back theorem. Therefore, $u \mapsto \sup_{a \in A} \sup_{v \in B} | \langle f(., a)_* u, v \rangle | = \sup_{v \in B'} | \langle u, v \rangle |$ is continuous in u since the right hand term is a continuous seminorm for the normal topology of $\mathcal{D}'_{\Gamma}(\Omega_1)$. □

Convolution product. In the same spirit as for the multiplication of distributions, the convolution product $u * v$ can be described as the composition of the tensor product $(u, v) \mapsto u \otimes v$ with the push–forward by the map $\Sigma := (x, y) \mapsto (x + y)$. For a closed subset $X \subset \mathbb{R}^n$, let $\mathcal{D}'_{\Gamma}(X)$ be the set of distributions supported in X with wave front in Γ. Therefore, we have the

Theorem 6.5. *Let Γ_1, Γ_2 be two closed conic sets in $\dot{T}^*\mathbb{R}^n$ and X_1, X_2 two closed subsets of $\mathbb{R}^n$ such that $\Sigma : X_1 \times X_2 \mapsto \mathbb{R}^n$ is proper. Then the convolution product of distributions is hypocontinuous from $\mathcal{D}'_{\Gamma_1}(X_1) \times \mathcal{D}'_{\Gamma_2}(X_2)$ to $\mathcal{D}'_{\Gamma}(X_1 + X_2)$ where*

$$\Gamma = \{(x + y; \eta) \, ; (x; \eta) \in \Gamma_1, (y; \eta) \in \Gamma_2\}. \tag{6.2}$$

Proof. The convolution product of distribution is the composition of the hypocontinuous tensor product with the continuous push–forward. □

As an application of the parameter version of the push-forward theorem, we can state the coordinate invariant definition of the wavefront set, which was proposed by Duistermaat [8, p. 13], correcting a first attempt by Gabor [9]. Its proof is left to the reader.

Theorem 6.6. *Let $\Omega \subset \mathbb{R}^d$ be an open set, $u \in \mathcal{D}'(\Omega)$. An element $(x; \xi) \notin WF_D(u)$ if and only if for all $f \in C^\infty(\Omega \times \mathbb{R}^n)$ such that $d_x f(x, a_0) = \xi$ for some $a_0 \in \mathbb{R}^n$, there exists some neighborhoods A of a_0 and U of x_0*

such that for all $\varphi \in \mathcal{D}(U)$: $|\left\langle u, \varphi e^{i\tau f(.,a)}\right\rangle| = O(\tau^{-\infty})$ uniformly in some neighborhood of a_0 in A.

7 Appendix: Technical results

This appendix gather different useful results. Several of them are folkore results for which we could find no proof in the literature.

7.1 Exhaustion of the complement of Γ

To prove that $\mathcal{D}'_\Gamma$ is nuclear [5], we need to take the additional seminorms in a countable set: the complement $\Gamma^c = \dot{T}^*M\backslash\Gamma$ of any closed cone $\Gamma \subset \dot{T}^*M$ can be exhausted by a countable set of products $U \times V$, where $U \subset M$ is compact and V is a closed conic subset of $\mathbb{R}^n$.

First we introduce the sphere bundle over M (or unit cotangent bundle) $UT^*M = \{(x;k) \in T^*M\,; |k| = 1\}$. We then define the set $U\Gamma_K = \{(x;k) \in \Gamma\,; x \in K, |k| = 1\} = UT^*M|_K \cap \Gamma$, for any compact subset $K \subset M$. Since M can be covered by a countable union of compact sets, we assume without loss of generality that K is covered by a single chart (U,ψ) such that $\psi(K) \subset Q$, where $Q = [-1,1]^n$ is an n-dimensional cube. We hence can assume w.l.g. that $M = \mathbb{R}^n$ and $UT^*M = \mathbb{R}^n \times S^{n-1}$. It will be convenient to use the norm d_∞ on $\mathbb{R}^n$, defined by: $\forall x,y \in \mathbb{R}^n$, $d_\infty(x,y) := \sup_{1\le i\le n} |x_i - y_i|$. We denote by $\overline{B}_\infty(x,r) = \{y \in \mathbb{R}^n\,; d_\infty(x,y) \le r\}$ the closed ball of radius r for this norm. We also denote the restriction of d_∞ to $S^{n-1}\times S^{n-1}$ by the same letter and, lastly, for $(x;\xi),(y;\eta) \in UT^*\mathbb{R}^n$ we set $d_\infty((x;\xi),(y;\eta)) = \sup(d_\infty(x,y), d_\infty(\xi,\eta))$.

We define cubes centered at rational points in Q: let $q_j = [-\frac{1}{2^j},\frac{1}{2^j}]^n = \overline{B}_\infty(0,2^{-j})$ and $q_{j,m} = \frac{m}{2^j} + q_j = \overline{B}_\infty(2^{-j}m, 2^{-j})$, where $m \in \mathbb{Z}^n \cap 2^jQ$. In other words, the center of $q_{j,m}$ runs over a hypercubic lattice with coordinates $(2^{-j}m_1,\dots,2^{-j}m_n)$, where $-2^j \le m_i \le 2^j$. Note that, for each non-negative integer j, the hypercubes $q_{j,m}$ overlap and cover Q:

$$Q \;\subset \bigcup_{m\in\mathbb{Z}^n\cap 2^jQ} q_{jm}. \tag{7.1}$$

Denote by $\underline{\pi} : UT^*\mathbb{R}^n \longrightarrow \mathbb{R}^n$ and $\overline{\pi} : UT^*\mathbb{R}^n \longrightarrow S^{n-1}$ the projection maps defined by $\underline{\pi}(x;k) = x$ and $\overline{\pi}(x;k) = k$. We define $F_{j,m} = \underline{\pi}^{-1}(q_{j,m}) \simeq q_{j,m}\times S^{n-1}$ (see Fig. 1). The set $\overline{\pi}(F_{j,m}\cap U\Gamma_K)$ is compact because the projection $\overline{\pi}$ is continuous and $F_{j,m}\cap U\Gamma_K$ is compact. For any positive integer ℓ, define the compact set $C_{j,m,\ell} = \big\{\eta \in S^{n-1}\,; d_\infty\big(\overline{\pi}(F_{j,m}\cap U\Gamma_K),\eta\big) \ge 1/\ell\big\}$.

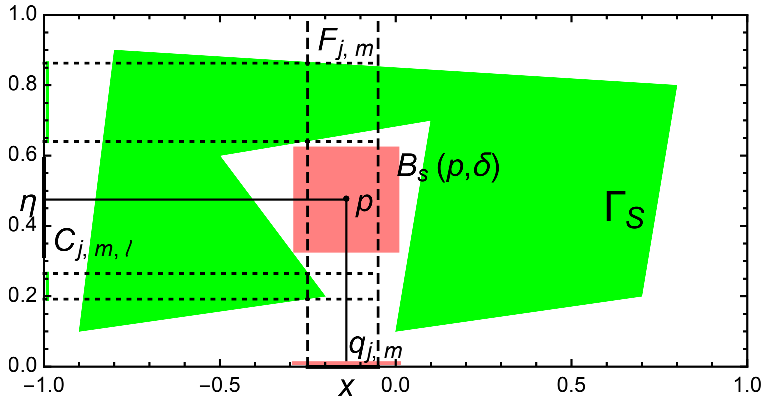

Figure 1: Picture of the case $n = 2$, where only one dimension of the cube $[-1,1]^2$ is shown and the circle S^1 is represented by the vertical segment $[0,1]$. The large surface is $U\Gamma_K$, the small square around p is the ball $\overline{B}_\infty(p,\delta)$. We see that $q_{j,m}$ contains x and is contained in $\underline{\pi}(\overline{B}_\infty(p,\delta))$.

This is the set of points of the sphere which are at least at a distance $1/\ell$ from the projection of the slice of $U\Gamma$ inside $F_{j,m}$ (see fig. 1). We have

$$\bigcup_{\ell>0} C_{j,m,\ell} \quad = \quad S^{n-1}\backslash\overline{\pi}(F_{j,m}\cap U\Gamma_K).$$

Indeed, by definition, any element of $C_{j,m,\ell}$ is in S^{n-1} and not in $\overline{\pi}(F_{j,m}\cap U\Gamma_K)$, conversely, the compactness of $\overline{\pi}(F_{j,m}\cap U\Gamma_K)$ implies that any point $(x;\xi)$ in $S^{n-1}\backslash\overline{\pi}(F_{j,m}\cap U\Gamma_K)$ is at a finite distance δ from $\overline{\pi}(F_{j,m}\cap U\Gamma_K)$. If we take $\ell > 1/\delta$, we have $(x;\xi)\in C_{j,m,\ell}$. Note that all $C_{j,m,\ell}$ are empty if $\overline{\pi}(F_{j,m}\cap U\Gamma_K) = S^{n-1}$. With this notation we can now state

Lemma 7.1.

$$\bigcup_{j,m,\ell} q_{j,m}\times C_{j,m,\ell} \quad = \quad UT^*M|_K\backslash U\Gamma_K, \tag{7.2}$$

and, by denoting $V_{j,m,\ell} = \{k\in\mathbb{R}^n\setminus\{0\}\,;k/|k|\in C_{j,m,\ell}\}$,

$$\bigcup_{j,m,\ell} q_{j,m}\times V_{j,m,\ell} \quad = \quad (T^*M\backslash\Gamma)|_K. \tag{7.3}$$

Proof. We first prove the inclusion $\subset$ in (7.2). Let $(x;k)\in\bigcup_{j,m,\ell} q_{j,m}\times C_{j,m,\ell}$, this means that there exist $j\in\mathbb{N}$, $m\in\mathbb{Z}^n\cap 2^jQ$ and $\ell\in\mathbb{N}^*$ such that $(x;k)\in q_{j,m}\times C_{j,m,\ell}$. Hence $(x;k)\in F_{j,m}$ and, by definition of $C_{j,m,\ell}$, $d_\infty(\overline{\pi}(F_{j,m}\cap U\Gamma_K),k)\geq 1/\ell$, which implies that $(x;k)\notin U\Gamma_K$.

Let us prove the reverse inclusion $\supset$. Let $(x;k) \in UT^*M|_K \setminus U\Gamma_K$. Since this set is open, there exists some $\delta > 0$ such that $\overline{B}_\infty((x;k),\delta) \subset UT^*M|_K \setminus U\Gamma_K$. Let $j \in \mathbb{N}^*$ s.t. $2^{-j+1} < \delta$. Because the sets $q_{j,m}$ cover Q (see eq. (7.1)), there is an m such that $x \in q_{jm}$. Moreover, $\forall y \in q_{j,m}$, we have $d_\infty(x,y) \leq d_\infty(x, 2^{-j}m) + d_\infty(2^{-j}m, y) \leq 2^{-j} + 2^{-j} < \delta$, i.e. $y \in \overline{B}_\infty(x,\delta)$. Hence $q_{j,m} \subset \overline{B}_\infty(x,\delta)$. We deduce that

$$q_{j,m} \times \overline{B}_\infty(k,\delta) \subset \overline{B}_\infty(x,\delta) \times \overline{B}_\infty(k,\delta) = \overline{B}_\infty((x;k),\delta) \subset UT^*M|_K \setminus U\Gamma_K.$$

This means that $(q_{j,m} \times \overline{B}_\infty(k,\delta)) \cap U\Gamma_K = \emptyset$ or equivalently $F_{j,m} \cap \overline{\pi}^{-1}(\overline{B}_\infty(k,\delta)) \cap U\Gamma_K = \emptyset$. The latter inclusion implies that $\overline{B}_\infty(k,\delta) \cap \overline{\pi}(F_{j,m} \cap U\Gamma_K) = \emptyset$. In other words, $d_\infty(k, \overline{\pi}(F_{j,m} \cap U\Gamma_K)) > \delta$. Hence by choosing $\ell \in \mathbb{N}^*$ s.t. $1/\ell \leq \delta$, we deduce that $d_\infty(k, \overline{\pi}(F_{j,m} \cap U\Gamma_K) > 1/\ell$, i.e. that $k \in C_{j,m,\ell}$. Thus we conclude that $(x;k) \in q_{j,m} \times C_{j,m,\ell}$ and Eq. (7.2) is proved.

To prove (7.3), we notice that, because of the conic property of Γ, each $C_{j,m,\ell}$ corresponds to a unique $V_{j,m,\ell}$. □

Any nonnegative smooth function ψ supported on $[-3/2, 3/2]^n$ and such that $\psi(x) = 1$ for $x \in [-1,1]^n$ enables us to define scaled and shifted functions $\psi_{j-1,m}(x) = \psi(2^j(x-m))$ supported on $q_{j-1,m}$ and equal to 1 on $q_{j,2m}$. If $C_{j,m,\ell}$ is not empty, we denote by $\alpha_{j,m,\ell} : S^{n-1} \to \mathbb{R}$ a smooth function supported on $C_{j,m,\ell}$ and equal to 1 on $C_{j,m,\ell+1}$. Note that, if $C_{j,m,\ell}$ is a proper subset of S^{n-1}, then it is strictly included in $C_{j,m,\ell+1}$.

7.2 Equivalence of topologies

Grigis and Sjöstrand stated [10, p. 80] that if we have a family χ_α of test functions and closed cones V_α such that $(\operatorname{supp}\chi_\alpha \times V_\alpha) \cap \Gamma = \emptyset$ and $\cup_\alpha \{(x,k)\,;\chi_\alpha(x) \neq 0 \text{ and } k \in \mathring{V}_\alpha\} = \Gamma^c$, then the topology of $\mathcal{D}'_\Gamma$ is the topology given by the seminorms of the weak topology and the seminorms $||\cdot||_{N,V_\alpha,\chi_\alpha}$. By covering M with a countable family of compact sets K_i described in section 7.1, we see that Lemma 7.1 gives us a family of indices $\alpha = (i,j,\ell)$, functions $\chi_{j,m,\ell} = \psi_{j,m}$ and cones $V_{j,m,\ell}$ adapted to K_i such that the conditions of the Grigis-Sjöstrand lemma are satisfied. Therefore, the normal topology is described by the seminorms of the strong topology of $\mathcal{D}'(\Omega)$ and by the countable family (i,j,m,ℓ) of seminorms.

7.3 Topological equivalence $C^\infty(X)$ and $\mathcal{D}'_\emptyset$

As an application of the previous lemma, we show

Lemma 7.2. *The spaces $C^\infty(X)$ and $\mathcal{D}'_\emptyset$ are topologically isomorphic.*

Proof. The two spaces are identical as vector spaces because a distribution u whose wave front set is empty is smooth everywhere, since its singular support $\operatorname{sing\,supp}(u) = \underline{\pi}(\mathrm{WF}(u))$ [12, p. 254] is empty [12, p. 42], and a distribution is a smooth function if and only if its singular support is empty.

To prove the topological equivalence, we must show that the two inclusions $\mathcal{D}'_\emptyset \hookrightarrow C^\infty(X)$ and $C^\infty(X) \hookrightarrow \mathcal{D}'_\emptyset$ are continuous. Recall that a system of semi-norms defining the topology of $C^\infty(X)$ is $\pi_{m,K}$, where m runs over the integers and K runs over the compact subsets of X [16, p. 88]. By a straightforward estimate, we obtain:

$$\begin{aligned}
\pi_{m,K}(\varphi) &\leq C_n(2\pi)^{-n} \sum_{|\alpha|\leq m} \sup_{k\in\mathbb{R}^n} (1+|k|^2)^p |k^\alpha \widehat{\varphi\chi}(k)| \\
&\leq C_n(2\pi)^{-n} \binom{m+n}{n} ||\varphi||_{m+2p,\mathbb{R}^n,\chi},
\end{aligned}$$

where $\chi \in \mathcal{D}(X)$ is equal to one on a compact set whose interior contains K where we used $(1+|k|^2) \leq (1+|k|)^2$, $|k^\alpha| \leq (1+|k|)^m$ and $\sum_{|\alpha|\leq m} = \binom{m+n}{n}$. Thus, every seminorm of $C^\infty(X)$ is bounded by a seminorm of $\mathcal{D}'_\emptyset$ and the injection $\mathcal{D}'_\emptyset \hookrightarrow C^\infty(X)$ is continuous.

Conversely, for any closed conic set V and any $\chi \in \mathcal{D}(X)$ we have $||\varphi||_{N,V,\chi} \leq ||\varphi||_{N,\mathbb{R}^n,\chi}$. Thus, it is enough to estimate $||\varphi||_{N,\mathbb{R}^n,\chi}$. We also find that, for any integer N and $\alpha = 0$, $\sup_{k\in\mathbb{R}^n}(1+k^2)^N|\widehat{\varphi\chi}(k)| \leq |K|2^N\pi_{2N,K}(\varphi\chi)$, where K is the support of φ. Then, the relation $1+|k| \leq 2(1+|k|^2)$ and application of the Leibniz rule give us $||\varphi||_{N,\mathbb{R}^n,\chi} \leq |K|8^N\pi_{2N,K}(\chi)\pi_{2N,K}(\varphi)$ and the seminorms $||\cdot||_{N,V,\chi}$ are controlled by the seminorms of $C^\infty(X)$. For the seminorms of $\mathcal{D}'(X)$, it is well known that the inclusion $C^\infty(X) \hookrightarrow \mathcal{D}'(X)$ is continuous [16, p. 420] when $\mathcal{D}'(X)$ has its strong topology. Therefore, it is also continuous when $\mathcal{D}'(X)$ has its weak topology and we proved that $C^\infty(X)$ and $\mathcal{D}'_\emptyset$ are topologically isomorphic, where $\mathcal{D}'_\emptyset$ can be equipped with the Hörmander or the normal topology. □

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